ON A THEOREM OF SHINTANI

CÉDRIC BONNAFÉ

Let $\mathbf{G}^\circ = \mathbf{GL}_n(\mathbb{F})^d$ where $\mathbb{F}$ is an algebraic closure of a finite field and where n and d are natural numbers. The symmetric group $\mathfrak{S}_d$ acts on $\mathbf{G}^\circ$ by permutations of the components of $\mathbf{G}^\circ$. We denote by $\mathbf{G}$ the semidirect product $\mathbf{G} = \mathbf{G}^\circ \rtimes \mathfrak{S}_d$. It is a non connected reductive group, with neutral component $\mathbf{G}^\circ$. We denote by $F_0 : \mathbf{G} \to \mathbf{G}$ the natural split Frobenius endomorphism on $\mathbf{G}$ (acting trivially on $\mathfrak{S}_d$) and we choose an element $\sigma \in \mathfrak{S}_d$. Let $F : \mathbf{G} \to \mathbf{G}$ denote the Frobenius endomorphism defined by $F(g) = {}^\sigma F_0(g)$.

In this paper we discuss the irreducible characters of $\mathbf{G}^F$ (in [B] were described the unipotent characters of $\mathbf{G}^F$). We first prove that there exists a Jordan decomposition of characters (*cf.* 1.7.2) as it is the case for $\mathbf{G}^\circ$: moreover, this decomposition commutes with Lusztig generalized induction (*cf.* 3.2.1). We also prove that all the irreducible characters of $\mathbf{G}^F$ are linear combinations of generalized Deligne-Lusztig characters (this generalizes the well-known result of G. Lusztig and B. Srinivasan [LS, theorem 3.2] about irreducible characters of the general linear group over a finite field) : this is proposition 2.3.2.

As an application of these results, we obtain new results about Shintani descent in the case of the general linear group. In [S], Shintani proved that any irreducible characters of the finite group $G_d = \mathbf{GL}_n(\mathbb{F}_{q^d})$ stable under the automorphism ϕ induced by the field automorphism $\mathbb{F}_{q^d} \to \mathbb{F}_{q^d}$, $x \mapsto x^q$ can be extended to $G_d\langle\phi\rangle$ in such a way that its Shintani descent is, up to sign, an irreducible character of $G_1 = \mathbf{GL}_n(\mathbb{F}_q)$. In theorem 4.3.1, we proved that this sign can always be chosen to be equal to 1 and get precise formulas for the corresponding extension. As a consequence, we obtain that this extension is compatible with different field extensions.

0. Notation

0.1. General notation. Let $\mathbb{F}$ be an algebraic closure of a finite field. We denote by p its characteristic. We also fix a power q of p and we denote by $\mathbb{F}_q$ the subfield of $\mathbb{F}$ with q elements. All algebraic varieties and all algebraic groups will be considered over $\mathbb{F}$. If $\mathbf{H}$ is an algebraic group (over $\mathbb{F}$), we will denote by $\mathbf{H}^\circ$ its connected component containing 1. If $\mathbf{H}$ is endowed with an $\mathbb{F}_q$-structure, we also define

$$\varepsilon_{\mathbf{H}^\circ} = (-1)^{\mathbb{F}_q-\mathrm{rank}(\mathbf{H}^\circ)}.$$

Let ℓ be a prime number different from p. We denote by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of the ℓ-adic field $\mathbb{Q}_\ell$. If G is a finite group, all representations and all characters of G will be considered over $\overline{\mathbb{Q}}_\ell$. For instance, a G-module is a $\overline{\mathbb{Q}}_\ell G$-module of finite dimension. We will denote by $\operatorname{Irr} G$ the set of irreducible characters of G.

If n is a positive integer, we denote by $\mathbf{GL}_n$ the group of invertible matrices with coefficients in $\mathbb{F}$, and if $g \in \mathbf{GL}_n$, we will denote by $g^{(q)}$ the matrix obtain from g by raising all coefficients

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to the q-th power. We will denote by $\mathbf{T}_n$ the split maximal torus of $\mathbf{GL}_n$ consisting of diagonal matrices and by $\mathbf{B}_n$ the rational Borel subgroup of $\mathbf{GL}_n$ consisting of upper triangular matrices.

0.2. The problem. Let r be a positive integer and let $d_1, \ldots, d_r$ and $n_1, \ldots, n_r$ be also positive integers. Throughout this paper $\mathbf{G}^\circ$ will denote the following connected reductive group :

$$\mathbf{G}^\circ = \prod_{i=1}^{r} (\underbrace{\mathbf{GL}_{n_i} \times \cdots \times \mathbf{GL}_{n_i}}_{d_i \text{ times}}).$$

We endow $\mathbf{G}^\circ$ with the split Frobenius endomorphism

$$\begin{array}{rccc} F_0 : & \mathbf{G}^\circ & \longrightarrow & \mathbf{G}^\circ \\ & (g_{i1}, \ldots, g_{id_i})_{1 \leqslant i \leqslant r} & \longmapsto & (g_{i1}^{(q)}, \ldots, g_{id_i}^{(q)})_{1 \leqslant i \leqslant r}. \end{array}$$

We will denote by $\mathbf{T}_0^\circ$ and $\mathbf{B}_0^\circ$ the maximal torus and the Borel subgroup of $\mathbf{G}^\circ$ defined respectively by

$$\mathbf{T}_0^\circ = \prod_{i=1}^{r} (\underbrace{\mathbf{T}_{n_i} \times \cdots \times \mathbf{T}_{n_i}}_{d_i \text{ times}})$$

and

$$\mathbf{B}_0^\circ = \prod_{i=1}^{r} (\underbrace{\mathbf{B}_{n_i} \times \cdots \times \mathbf{B}_{n_i}}_{d_i \text{ times}}).$$

The group $\mathfrak{S} = \mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_r}$ acts on $\mathbf{G}^\circ$ in the natural way. More explicitly, if $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathfrak{S}$ and if $(g_{i1}, \ldots, g_{id_i})_{1 \leqslant i \leqslant r} \in \mathbf{G}^\circ$, we put

$$^{\sigma}(g_{i1}, \ldots, g_{id_i})_{1 \leqslant i \leqslant r} = (g_{i\sigma_i^{-1}(1)}, \ldots, g_{i\sigma_i^{-1}(d_i)})_{1 \leqslant i \leqslant r}.$$

The elements of $\mathfrak{S}$ induces automorphism of $\mathbf{G}^\circ$ which stabilize $\mathbf{T}_0^\circ$ and $\mathbf{B}_0^\circ$, so they are quasi-semisimple (*cf.* [DM2], definition 1.1, (i)). In fact, they are all quasi-central (*cf.* [DM2], definition-theorem 1.15 and [B], lemma 7.1.1).

We extend the Frobenius endomorphism F_0 to $\mathbf{G}^\circ \rtimes \mathfrak{S}$by letting F_0 act trivially on $\mathfrak{S}$. We fix once and for all an element $\sigma \in \mathfrak{S}$ and we denote by F the Frobenius endomorphism on $\mathbf{G}^\circ \rtimes \mathfrak{S}$ given by

$$F(g) = \sigma F_0(g) \sigma^{-1} = {}^{\sigma}F_0(g)$$

for all $g \in \mathbf{G}^\circ \rtimes \mathfrak{S}$.

We will denote by $\mathbf{G}$ an F-stable subgroup of $\mathbf{G}^\circ \rtimes \mathfrak{S}$ containing $\mathbf{G}^\circ$. Hence $\mathbf{G}$ is a reductive group with neutral component $\mathbf{G}^\circ$. Moreover, there exists an F-stable (that is, a σ-stable) subgroup A of $\mathfrak{S}$ such that

$$\mathbf{G} = \mathbf{G}^\circ \rtimes A.$$

Thus we have $\mathbf{G}^F = \mathbf{G}^{\circ F} \rtimes A^F = \mathbf{G}^{\circ F} \rtimes A^\sigma$.

Problem : *Parametrize the irreducible characters of* $\mathbf{G}^F$*.*

For this purpose we can make the following hypothesis without loss of generality :

Hypothesis : *The Frobenius endomorphism F acts trivially on $\mathbf{G}/\mathbf{G}^\circ$, that is, A is contained in the centralizer of σ in $\mathfrak{S}$. Consequently,*

$$\mathbf{G}^F = \mathbf{G}^{\circ F} \rtimes A.$$

Remark 0 : Let $N = d_1n_1+\cdots+d_rn_r$. Then $\mathbf{G}^\circ$ is isomorphic to a rational Levi subgroup $\mathbf{H}^\circ$ of a parabolic subgroup of $\mathbf{GL}_N$ (endowed with the split Frobenius endomorphism $g \mapsto g^{(q)}$), and $\mathbf{G}$ is isomorphic to a rational subgroup $\mathbf{H}$ of the normalizer of $\mathbf{H}^\circ$ in $\mathbf{GL}_N$, containing $\mathbf{H}^\circ$ and such that all elements of $\mathbf{H}/\mathbf{H}^\circ$ are rational. Conversely, if $\mathbf{H}$ is such a rational subgroup of $\mathbf{GL}_N$, then there exist positive integers r, d_1, ... , d_r, n_1, ... , n_r ; an element σ of $\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_r}$; and a subgroup A of $\mathfrak{S}^\sigma$ such that $\mathbf{H}$ is isomorphic to the group $\mathbf{G}$ constructed as above. *In particular, if $\mathbf{L}$ is an F-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}$ (cf. [B], definitions 6.1.1 and 6.1.2 for the definitions of parabolic subgroups and Levi subgroups of non-connected reductive groups), then all the results proved for $\mathbf{G}$ hold in $\mathbf{L}$.*

1. JORDAN DECOMPOSITION OF CHARACTERS OF $\mathbf{G}^F$

1.1. Dual of G. Let $(\mathbf{G}^{\circ *}, \mathbf{T}_0^{\circ *}, F^*)$ be a dual triple of $(\mathbf{G}^\circ, \mathbf{T}_0^\circ, F)$. The elements α of $\mathfrak{S}$ induce automorphisms α^* of $\mathbf{G}^{\circ *}$. The group $\mathfrak{S}^*$ of automorphisms of $\mathbf{G}^{\circ *}$ induced by $\mathfrak{S}$ is isomorphic to the opposite group of $\mathfrak{S}$. We extend the action of F^* to $\mathbf{G}^{\circ *} \rtimes \mathfrak{S}^*$ so that it acts on $\mathfrak{S}^*$ by conjugation by σ^{*-1}. We denote by $\mathbf{G}^*$ the semi-direct product $\mathbf{G}^{\circ *} \rtimes A^*$ where A^* is the image of A under the preceding anti-isomorphism. In particular, $\mathbf{G}^{*\circ} = \mathbf{G}^{\circ *}$!

1.2. Lusztig series of $\mathbf{G}^F$. Let s be a semi-simple element of $\mathbf{G}^{*\circ F^*}$. We denote by (s) (or $(s)_{\mathbf{G}^{*F^*}}$ if confusion is possible) the $\mathbf{G}^{*F^*}$-conjugacy class of s and by $(s)^\circ$ (or $(s)^\circ_{\mathbf{G}^{*\circ F^*}}$) the $\mathbf{G}^{*\circ F^*}$-conjugacy class of s.

Definition 1.2.1. *The* **Lusztig series** *$\mathcal{E}(\mathbf{G}^F, (s))$ of $\mathbf{G}^F$ associated to s (or (s)) is the set of irreducible characters of $\mathbf{G}^F$ occuring in some $\operatorname{Ind}_{\mathbf{G}^{\circ F}}^{\mathbf{G}^F} \gamma^\circ$, where γ° is an element of a usual Lusztig series $\mathcal{E}(\mathbf{G}^{\circ F}, (s')^\circ)$ with $s' \in (s)$.*

The characters of the Lusztig series $\mathcal{E}(\mathbf{G}^F, 1)$ are called **unipotent** *: this definition agrees with definitions given in [DM2], section 5 or [B], definition 6.4.1 (cf. [B], lemma 6.4.2).*

The following lemma follows immediately from the definitions :

Lemma 1.2.2. *Let s be a semisimple element of $\mathbf{G}^{*\circ F^*}$, γ° be an element of $\mathcal{E}(\mathbf{G}^{\circ F}, (s)^\circ)$, and $\alpha \in A$. Then ${}^\alpha\gamma^\circ \in \mathcal{E}(\mathbf{G}^{\circ F}, ({}^{\alpha^{*-1}}s)^\circ)$.*

Corollary 1.2.3.

$$\operatorname{Irr} \mathbf{G}^F = \bigcup_{(s)} \mathcal{E}(\mathbf{G}^F, (s))$$

where (s) runs over the set of $\mathbf{G}^{*F^*}$-classes of semisimple elements of $\mathbf{G}^{*\circ F^*}$. Moreover, this union is disjoint.

PROOF - The equality follows easily from the corresponding fact for $\mathbf{G}^{\circ F}$. Let us prove now that the union is disjoint. Let s and t be two semisimple elements of $\mathbf{G}^{*\circ F^*}$ and let γ be an irreducible character of $\mathbf{G}^F$ belonging to both $\mathcal{E}(\mathbf{G}^F, (s))$ and $\mathcal{E}(\mathbf{G}^F, (t))$. Then by definition there exist irreducible characters γ_1° and γ_2° of $\mathbf{G}^{\circ F}$ occuring in the restriction of γ to $\mathbf{G}^{\circ F}$ such that $\gamma_1^\circ \in \mathcal{E}(\mathbf{G}^{\circ F}, (s')^\circ)$ and $\gamma_2^\circ \in \mathcal{E}(\mathbf{G}^{\circ F}, (t')^\circ)$ where $s' \in (s)$ and $t' \in (t)$.

But by Clifford theory there exists $\alpha \in A$ such that $\gamma_2^\circ = {}^\alpha\gamma_1^\circ$. It follows from lemma 1.2.2 and from the fact that the corollary 1.2.3 holds in $\mathbf{G}^\circ$ that $t' \in ({}^{\alpha^{*-1}}s')^\circ$ so $t \in (s)$. ■

Corollary 1.2.4. *Let s be a semisimple element in $\mathbf{G}^{*\circ F^*}$ and let $\gamma \in \mathcal{E}(\mathbf{G}^F, (s))$.*

(a) *Let γ° be an irreducible component of the restriction of γ to $\mathbf{G}^{\circ F}$ and let t be a semisimple element of $\mathbf{G}^{*\circ F^*}$ such that $\gamma^\circ \in \mathcal{E}(\mathbf{G}^{\circ F}, (t)^\circ)$. Then $t \in (s)$.*

(b) *There exists an irreducible component of the restriction of γ to $\mathbf{G}^{\circ F}$ belonging to $\mathcal{E}(\mathbf{G}^{\circ F}, (s)^\circ)$.*

Proof - (a) is a reformulation of the corollary 1.2.3 and (b) is an easy consequence of (a) and of the lemma 1.2.2. ■

1.3. Nice elements. Let s be a semisimple element of $\mathbf{G}^{*\circ F^*}$. The centralizer of s in $\mathbf{G}^{*\circ}$ is connected and is a Levi subgroup of a parabolic subgroup of $\mathbf{G}^{*\circ}$. The image of $C_{\mathbf{G}^*}(s)$ by the morphism

$$C_{\mathbf{G}^*}(s) \longrightarrow \mathbf{G}^* \longrightarrow A^*$$

is denoted by $A^*(s)$. Then the $\mathbf{G}^{*\circ}$-conjugacy class of s is stable under $A^*(s)$. If we denote by $\mathbf{G}^{*\circ A^*(s)}$ the group of fixed points of $A^*(s)$ on $\mathbf{G}^{*\circ}$, then the $\mathbf{G}^{*\circ}$-conjugacy class of s in $\mathbf{G}^{*\circ}$ meets $\mathbf{G}^{*\circ A^*(s)}$ in a single $\mathbf{G}^{*\circ A^*(s)}$-conjugacy class because $A^*(s)$ acts by permutations on the component of $\mathbf{G}^{*\circ}$. This conjugacy class is F^*-stable and $\mathbf{G}^{*\circ A^*(s)}$ is connected, so there exists an F^*-stable element t in the $\mathbf{G}^{*\circ}$-conjugacy class of s centralized by $A^*(s)$. Moreover $C_{\mathbf{G}^{*\circ}}(s)$ is connected so $t \in (s)^\circ$. It also implies that $A^*(t)$ contains $A^*(s)$. Because they are conjugate under A^*, they are equal.

Definition 1.3.1. *The element s is said to be* **nice** *(or* **$\mathbf{G}^*$-nice***) if $A^*(s)$ centralizes s.*

The preceding discussion shows that there exists a nice element in every semisimple $\mathbf{G}^{*\circ F^*}$-conjugacy class. If s is a nice element of $\mathbf{G}^{*\circ F^*}$ and if $\alpha^* \in A^*$ is such that ${}^{\alpha^*}(s)^\circ = (s)^\circ$, then $\alpha^* \in A^*(s)$.

1.4. The group $\mathbf{G}(s)$. Until the end of this section, we fix a nice semisimple element s in $\mathbf{G}^{*\circ F^*}$. Let $A(s)$ be the subgroup of A corresponding to $A^*(s)$. The group $C_{\mathbf{G}^{*\circ}}(s) = C_{\mathbf{G}^*}(s)^\circ$ is an F^*-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^{*\circ}$. Let $\mathbf{G}^\circ(s)$ be an F-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^\circ$ dual to $C_{\mathbf{G}^{*\circ}}(s)$; we can assume that $A(s)$ normalizes $\mathbf{G}^\circ(s)$. We define $\mathbf{G}(s)$ to be the semidirect product

$$\mathbf{(1.4.1)} \qquad \mathbf{G}(s) = \mathbf{G}^\circ(s) \rtimes A(s).$$

Because $A(s)$ acts on $\mathbf{G}^\circ$ by permutations of the components, there exists a parabolic subgroup of $\mathbf{G}^\circ$ having $\mathbf{G}^\circ(s)$ as a Levi subgroup and stable under $A(s)$. Hence, $\mathbf{G}(s)$ is a Levi subgroup of a parabolic subgroup of $\mathbf{G}$. Moreover, $\mathbf{G}(s)^\circ = \mathbf{G}^\circ(s)$.

To the semisimple element s is associated a linear character $\hat{s}^\circ$ of $\mathbf{G}^\circ(s)^F$ (*cf.* [DM1], proposition 13.30). Because s is centralized by $A^*(s)$, the character $\hat{s}^\circ$ is invariant by $A(s)$, so it extends to a character $\hat{s}$ of $\mathbf{G}(s)^F$, where $\hat{s}(\alpha) = 1$ for $\alpha \in A(s)$.

1.5. A lemma. Let $\gamma^\circ(s)$ be a unipotent character of $\mathbf{G}^\circ(s)^F$. By [B], theorem 7.3.2 and definition 7.3.3, there exists a canonical extension $\tilde{\gamma}(s)$ of $\gamma^\circ(s)$ to $\mathbf{G}^\circ(s)^F \rtimes A(s, \gamma^\circ(s))$ where $A(s, \gamma^\circ(s))$ is the stabilizer of $\gamma^\circ(s)$ in $A(s)$.

Lemma 1.5.1. *$\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R_{\mathbf{G}^\circ(s)\rtimes A(s,\gamma^\circ(s))}^{\mathbf{G}^\circ \rtimes A(s,\gamma^\circ(s))}(\tilde{\gamma}(s) \otimes \hat{s})$ is an irreducible character of the group $\mathbf{G}^{\circ F} \rtimes A(s, \gamma^\circ(s))$. Its restriction to $\mathbf{G}^{\circ F}$ is the irreducible character $\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R_{\mathbf{G}^\circ(s)}^{\mathbf{G}^\circ}(\gamma^\circ(s) \otimes \hat{s}^\circ)$ in $\mathcal{E}(\mathbf{G}^{\circ F}, (s)^\circ)$.*

Remark - By [B], theorem 7.3.2, the unipotent character $\tilde{\gamma}(s)$ of $\mathbf{G}^\circ(s)^F \rtimes A(s,\gamma^\circ(s))$ is a uniform function, that is, a linear combination of generalized Deligne-Lusztig characters. Hence the class function $\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}^\circ \rtimes A(s,\gamma^\circ(s))}_{\mathbf{G}^\circ(s) \rtimes A(s,\gamma^\circ(s))}(\tilde{\gamma}(s) \otimes \hat{s})$ is independent of the choice of a parabolic subgroup of $\mathbf{G}$ having $\mathbf{G}^\circ(s) \rtimes A(s,\gamma^\circ(s))$ as Levi subgroup. That is the reason why the Lusztig functor is denoted without reference to the parabolic subgroup (the notion of Lusztig functor for disconnected reductive groups has been defined in DM2, and slightly generalized for the purpose of this article in [B]).

Proof of lemma 1.5.1 - To simplify notation, we can assume that $A = A(s,\gamma^\circ(s))$.

Let
$$\tilde{\gamma} = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}}_{\mathbf{G}(s)}(\tilde{\gamma}(s) \otimes \hat{s})$$

and
$$\gamma^\circ = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}^\circ}_{\mathbf{G}^\circ(s)}(\gamma^\circ(s) \otimes \hat{s}^\circ).$$

It results from [DM2], corollary 2.4, that the restriction of $\tilde{\gamma}$ to $\mathbf{G}^{\circ F}$ is equal to γ°. Moreover, by [LS], theorem 3.2, γ° is irreducible and lies in $\mathcal{E}(\mathbf{G}^{\circ F},(s)^\circ)$. So we need only prove that $\tilde{\gamma}$ is a character of $\mathbf{G}^F$.

Let $\mathbf{P}(s)$ be a parabolic subgroup of $\mathbf{G}(s)$ having $\mathbf{G}(s)$ as Levi subgroup and let $\mathbf{U}$ be its unipotent radical. We define :

$$\mathbf{Y}_\mathbf{U} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}$$

and
$$\mathbf{Y}^\circ_\mathbf{U} = \{g \in \mathbf{G}^\circ \mid g^{-1}F(g) \in \mathbf{U}\}.$$

Let $H^i_c(\mathbf{Y}_\mathbf{U})$ be the i-th cohomology group with compact support with coefficients in the constant sheaf $\overline{\mathbb{Q}}_\ell$ (where $i \in \mathbb{N}$). The group $\mathbf{G}^F$ (respectively $\mathbf{G}(s)^F$) acts on $\mathbf{Y}_\mathbf{U}$ by left (respectively right) translation. Hence $H^i_c(\mathbf{Y}_\mathbf{U})$ inherits the structure of an $\mathbf{G}^F$-module-$\mathbf{G}(s)^F$. Let V be an irreducible $\mathbf{G}(s)^F$-module affording $\tilde{\gamma}(s)$ as character. Then the virtual $\mathbf{G}^F$-module

$$\sum_{i \in \mathbb{N}} (-1)^i \, H^i_c(\mathbf{Y}_\mathbf{U}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} V$$

affords $\tilde{\gamma}$ as (virtual) character. We have similar results for $\mathbf{G}^{\circ F}$. We denote by V° the restriction of V to $\mathbf{G}^{\circ F}$.

By [DM1], , theorem 13.25, (i), there exists j in $\mathbb{N}$ such that

$$H^i_c(\mathbf{Y}^\circ_\mathbf{U}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ = 0$$

if $i \neq j$ and such that

$$H^j_c(\mathbf{Y}^\circ_\mathbf{U}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ$$

is irreducible (in DM1, the statement and the proof of theorem 13.25 are not entirely correct : a precise value for j is given and it is not clear that this value is correct. However, the existence of j satisfying the above conditions has been established in a revised version of their book). Moreover, $(-1)^j = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ}$. But by [DM2], proof of proposition 2.3, we have

$$H^i_c(\mathbf{Y}_\mathbf{U}) = \overline{\mathbb{Q}}_\ell \mathbf{G}^F \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^{\circ F}} H^i_c(\mathbf{Y}^\circ_\mathbf{U})$$

as a $\mathbf{G}^F$-module. Hence we have

$$H^i_c(\mathbf{Y}_\mathbf{U}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ = 0$$

for all $i \neq j$. But

$$\begin{aligned} H^i_c(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ &= \Big(H^i_c(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} \overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F \Big) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ \\ &= H^i_c(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} \Big(\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}^\circ(s)^F} V^\circ \Big) \\ &= H^i_c(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} \operatorname{Ind}_{\mathbf{G}^\circ(s)^F}^{\mathbf{G}(s)^F} V^\circ. \end{aligned}$$

Because V is a direct summand of the $\mathbf{G}(s)^F$-module $\operatorname{Ind}_{\mathbf{G}^\circ(s)^F}^{\mathbf{G}(s)^F} V^\circ$, it follows that

$$H^i_c(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} V = 0$$

if $i \neq j$ and that $\tilde{\gamma}$ is the character of the module

$$H^j_c(\mathbf{Y}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{G}(s)^F} V. \blacksquare$$

1.6. Clifford theory. Let $\gamma^\circ \in \mathcal{E}(\mathbf{G}^{\circ F}, (s)^\circ)$. By [LS], theorem 3.2, there exists a unique unipotent character $\gamma^\circ(s)$ of $\mathbf{G}^\circ(s)^F$ such that

(1.6.1)
$$\gamma^\circ = \varepsilon_{\mathbf{G}^\circ(s)} \varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}^\circ}_{\mathbf{G}^\circ(s)}(\gamma^\circ(s) \otimes \hat{s}^\circ).$$

Let $A(\gamma^\circ)$ be the stabilizer of γ° in A. Its dual $A^*(\gamma^\circ)$ stabilizes the $\mathbf{G}^{*\circ F^*}$-conjugacy class of s and hence is contained in $A^*(s)$. By duality $A(\gamma^\circ)$ is contained in $A(s)$. The uniqueness of $\gamma^\circ(s)$ implies that $A(\gamma^\circ)$ is the stabilizer $A(s, \gamma^\circ(s))$ of $\gamma^\circ(s)$ in $A(s)$.

We denote by $\tilde{\gamma}(s)$ the canonical extension of $\gamma^\circ(s)$ to $\mathbf{G}^\circ(s)^F \rtimes A(\gamma^\circ)$ (as defined in [B], definition 7.3.3). We put :

(1.6.2)
$$\tilde{\gamma} = \varepsilon_{\mathbf{G}^\circ(s)} \varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}^\circ \rtimes A(\gamma^\circ)}_{\mathbf{G}^\circ(s) \rtimes A(\gamma^\circ)}(\tilde{\gamma}(s) \otimes \hat{s}).$$

Then, by lemma 1.5.1, $\tilde{\gamma}$ is an irreducible character of $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ extending γ°.

Definition 1.6.3. *The irreducible character* $\tilde{\gamma}$ *of* $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ *will be called the* **canonical** *extension of* γ°*.*

If ξ is an irreducible character of $A(\gamma^\circ)$, then by Clifford theory $\tilde{\gamma} \otimes \xi$ is an irreducible character of $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ and $\operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)}(\tilde{\gamma} \otimes \xi)$ is an irreducible character of $\mathbf{G}^F$ (where ξ is lifted to $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ in the natural way). Moreover,

(1.6.4)
$$\operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F}} \gamma^\circ = \sum_{\xi \in \operatorname{Irr} A(\gamma^\circ)} \xi(1) \ \operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)}(\tilde{\gamma} \otimes \xi).$$

1.7. Jordan decomposition. Let γ be an irreducible character in $\mathcal{E}(\mathbf{G}^F, (s))$. By corollary 1.2.4 there exists an irreducible character $\gamma^\circ \in \mathcal{E}(\mathbf{G}^{\circ F}, (s)^\circ)$ occuring in the restriction of γ to $\mathbf{G}^{\circ F}$. Let $\tilde{\gamma}$ be the canonical extension of γ° to $\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)$ defined in 1.6.3. Then by Clifford theory there exists a unique irreducible character ξ of $A(\gamma^\circ)$ such that

$$\gamma = \operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F} \rtimes A(\gamma^\circ)}(\tilde{\gamma} \otimes \xi).$$

Let $\gamma^\circ(s)$ be the unipotent character of $\mathbf{G}^\circ(s)^F$ satisfying 1.6.1 and let $\tilde{\gamma}(s)$ be its canonical extension to $\mathbf{G}^\circ(s)^F \rtimes A(\gamma^\circ)$ (recall that $A(\gamma^\circ)$ is the stabilizer of $\gamma^\circ(s)$ in $A(s)$). Then

$$\gamma(s) = \operatorname{Ind}^{\mathbf{G}(s)^F}_{\mathbf{G}^\circ(s)^F \rtimes A(\gamma^\circ)}(\tilde{\gamma}(s) \otimes \xi)$$

is an irreducible character of $\mathbf{G}(s)^F$ and is unipotent by definition. It follows from [B], propositions 6.3.2 and 6.3.3 that :

$$\gamma = \varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ} R^{\mathbf{G}}_{\mathbf{G}(s)}(\gamma(s)\otimes\hat{s}). \tag{1.7.1}$$

Remark - The remark following lemma 1.5.1 shows that the Lusztig functor $R^{\mathbf{G}}_{\mathbf{G}(s)}$ does not depend on the choice of a parabolic subgroup of $\mathbf{G}$ having $\mathbf{G}(s)$ as Levi subgroup.

Jordan decomposition of irreducible characters. *With the above notation the map*

$$\begin{array}{rccc} \nabla_{\mathbf{G},s}: & \mathcal{E}(\mathbf{G}^F,(s)) & \longrightarrow & \mathcal{E}(\mathbf{G}(s)^F,1) \\ & \gamma & \longmapsto & \gamma(s) \end{array} \tag{1.7.2}$$

is well-defined and bijective. The inverse map is given by formula 1.7.1.

Proof - First we have to prove that $\nabla_{\mathbf{G},s}$ is well-defined. There is one ambiguity in the construction of $\gamma(s)$: in the first step, we chose an irreducible character $\gamma^\circ \in \mathcal{E}(\mathbf{G}^{\circ F},(s)^\circ)$ occuring in the restriction of γ to $\mathbf{G}^{\circ F}$. If δ° is another element of the Lusztig series $\mathcal{E}(\mathbf{G}^{\circ F},(s)^\circ)$ occuring in the restriction of γ to $\mathbf{G}^{\circ F}$, then there exists $\alpha \in A$ such that $\delta^\circ = {}^\alpha\gamma^\circ$. But both lie in $\mathcal{E}(\mathbf{G}^{\circ F},(s)^\circ)$, so we have $\alpha \in A(s)$. If we construct $\delta^\circ(s)$, $\tilde{\delta}(s)$ and $\delta(s)$ in the same way as $\gamma^\circ(s)$, $\tilde{\gamma}(s)$ and $\gamma(s)$ respectively, then $\delta^\circ(s) = {}^\alpha\gamma^\circ(s)$ (by uniqueness), so $\tilde{\delta}(s) = {}^\alpha\tilde{\gamma}(s)$ and so $\delta(s) = {}^\alpha\gamma(s) = \gamma(s)$ because $\alpha \in A(s)$. Thus $\nabla_{\mathbf{G},s}$ is well-defined.

$\nabla_{\mathbf{G},s}$ is injective by formula 1.7.1 and surjective by lemma 1.5.1 which proves that formula 1.7.1 always defines an element of $\mathcal{E}(\mathbf{G}^F,(s))$. ■

2. Uniform functions

In [B], formula 7.3.1 and theorem 7.3.2, the unipotent characters of $\mathbf{G}^F$ are described as linear combinations of generalized Deligne-Lusztig characters. It is possible using formula 1.7.1 to describe all the irreducible characters of $\mathbf{G}^F$ as linear combinations of generalized Deligne-Lusztig characters.

2.1. Notation. Let s be a nice semisimple element of $\mathbf{G}^{*\circ F^*}$.

We fix an F-stable and $A(s)$-stable Borel subgroup $\mathbf{B}_1^\circ(s)$ of $\mathbf{G}^\circ(s)$ and an F-stable and $A(s)$-stable maximal torus $\mathbf{T}_1^\circ(s)$ of $\mathbf{B}_1^\circ(s)$. We denote by $W(s)$ (respectively $W^\circ(s)$) the Weyl group of $\mathbf{G}(s)$ (respectively $\mathbf{G}^\circ(s)$) relative to $\mathbf{T}_1^\circ(s)$.

For each $\alpha \in A(s)$, we define $\mathbf{T}_1^\circ(s,\alpha)$ to be the semidirect product $\mathbf{T}_1^\circ(s) \rtimes \langle\alpha\rangle$. For each $w \in W^\circ(s)^\alpha$ (that is, the subgroup of $W^\circ(s)$ consisting of elements centralized by α), we denote by $\mathbf{T}_w(s,\alpha)$ the **quasi-maximal torus** of $\mathbf{G}^\circ(s) \rtimes \langle\alpha\rangle$ associated to w as in [DM2], proposition 1.40 (for the definition of a quasi-maximal torus, *cf.* [B], definition 6.1.3). $\mathbf{T}_w(s,\alpha)$ is defined by the following property : $(\mathbf{T}_w(s,\alpha)^\alpha)^\circ$ is an F-stable maximal torus of $\mathbf{G}^\circ(s)^\alpha$ of type w with respect to $\mathbf{T}_1^\circ(s)^\alpha$.

The group $W^\circ(s)$ is a product of symmetric groups, and $A(s)$ and F act on $W^\circ(s)$ by permutations of the components (F acts on $W^\circ(s)$ as σ). By the argument used in [B], section 7.3, we can associate canonically to each irreducible character χ° of $W^\circ(s)^F$ and each α in the stabilizer $A(s,\chi^\circ)$ of χ° in $A(s)$ an irreducible character $\tilde{\boldsymbol{\chi}}_\alpha$ of $W^\circ(s)^\alpha \rtimes \langle\sigma\rangle$.

2.2. Irreducible characters in $\mathcal{E}(\mathbf{G}^{\circ F},(s)^\circ)$ as uniform functions. Let χ° be an irreducible character of $W^\circ(s)^F$. We define

$$R^\circ_{\chi^\circ}(s) = R^{\mathbf{G}^\circ}_{\chi^\circ}(s) = \frac{\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ}}{|W^\circ(s)|} \sum_{w \in W^\circ(s)} \tilde{\boldsymbol{\chi}}_1(w\sigma)\ R^{\mathbf{G}^\circ}_{\mathbf{T}_{w(s,1)}}(\hat{s}^\circ). \tag{2.2.1}$$

Proposition 2.2.2 (Lusztig-Srinivasan, theorem 3.2)**.** (a) *For all $\chi^\circ \in \operatorname{Irr} W^\circ(s)^F$, $R^\circ_{\chi^\circ}(s)$ is an irreducible character of $\mathbf{G}^{\circ F}$ in $\mathcal{E}(\mathbf{G}^{\circ F},(s)^\circ)$.*

(b) *The map*

$$\begin{array}{ccc} \operatorname{Irr} W^\circ(s)^F & \longrightarrow & \mathcal{E}(\mathbf{G}^{\circ F},(s)^\circ) \\ \chi^\circ & \longmapsto & R^\circ_{\chi^\circ}(s) \end{array}$$

is bijective.

Corollary 2.2.3. (a) *If $\chi^\circ \in \operatorname{Irr} W^\circ(s)^F$ and $\alpha \in A(s)$, then ${}^\alpha R^\circ_{\chi^\circ}(s) = R^\circ_{{}^\alpha\chi^\circ}(s)$.*

(b) *If $\chi^\circ \in \operatorname{Irr} W^\circ(s)^F$, then $A(R^\circ_{\chi^\circ}(s)) = A(s,\chi^\circ)$.*

2.3. Canonical extensions as uniform functions. Let χ° be an irreducible character of $W^\circ(s)^F$. We define a function $\tilde{R}_{\chi^\circ}(s)$ on $\mathbf{G}^{\circ F} \rtimes A(s,\chi^\circ)$ by

$$\operatorname{Res}^{\mathbf{G}^{\circ F} \rtimes A(s,\chi^\circ)}_{\mathbf{G}^{\circ F}.\alpha} \tilde{R}_{\chi^\circ}(s) = \frac{\varepsilon_{\mathbf{G}^\circ(s)}\varepsilon_{\mathbf{G}^\circ}}{|W^\circ(s)^\alpha|} \sum_{w \in W^\circ(s)^\alpha} \tilde{\boldsymbol{\chi}}_\alpha(w\sigma)\ \operatorname{Res}^{\mathbf{G}^{\circ F} \rtimes \langle\alpha\rangle}_{\mathbf{G}^{\circ F}.\alpha} R^{\mathbf{G}^\circ \rtimes \langle\alpha\rangle}_{\mathbf{T}_{w(s,\alpha)}}(\hat{s}) \tag{2.3.1}$$

for all $\alpha \in A(s,\chi^\circ)$.

Proposition 2.3.2. *$\tilde{R}_{\chi^\circ}(s)$ is an irreducible character of $\mathbf{G}^{\circ F} \rtimes A(s,\chi^\circ)$ and is in fact the canonical extension of $R^\circ_{\chi^\circ}(s)$ (cf. definition 1.6.3).*

Proof - This follows immediately from formula 1.6.2, from [B], theorem 7.3.2 and from [DM2], proposition 2.3. ∎

2.4. Parametrization of $\mathcal{E}(\mathbf{G}^F,(s))$. We denote by $\mathcal{I}(s)$ the set of pairs (χ°,ξ) where χ° is an irreducible character of $W^\circ(s)^F$ and ξ is an irreducible character of $A(s,\chi^\circ)$. The group $A(s)$ acts by conjugation on $\mathcal{I}(s)$, and we denote by $\bar{\mathcal{I}}(s)$ the set of orbits of $A(s)$ in $\mathcal{I}(s)$. Moreover, if $(\chi^\circ,\xi) \in \mathcal{I}(s)$, we denote by $\chi^\circ * \xi$ its orbit under $A(s)$.

For all $\chi^\circ * \xi \in \bar{\mathcal{I}}(s)$, we define

$$R^{\mathbf{G}}_{\chi^\circ * \xi}(s) = R_{\chi^\circ * \xi}(s) = \operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{G}^{\circ F} \rtimes A(s,\chi^\circ)}(\tilde{R}_{\chi^\circ}(s) \otimes \xi). \tag{2.4.1}$$

It follows from corollary 2.2.3, (a), that $R_{\chi^\circ * \xi}(s)$ only depends on the orbit of (χ°,ξ) under $A(s)$. Moreover, it follows from Clifford theory and from corollary 2.2.3, (b), that we have the

Lemma 2.4.2. *The map*

$$\begin{array}{ccc} \bar{\mathcal{I}}(s) & \longrightarrow & \mathcal{E}(\mathbf{G}^F,(s)) \\ \chi^\circ * \xi & \longmapsto & R_{\chi^\circ * \xi}(s) \end{array}$$

is bijective.

By [B], proposition 2.3.1, χ° has a canonical extension $\tilde{\chi}$ to the semidirect product $W^\circ(s) \rtimes A(s,\chi^\circ)$. By Clifford theory again we have the

Lemma 2.4.3. *The map*

$$\begin{array}{ccc} \bar{\mathcal{I}}(s) & \longrightarrow & \operatorname{Irr} W(s)^F \\ \chi^\circ * \xi & \longmapsto & \operatorname{Ind}_{W^\circ(s)^F \rtimes A(s,\chi^\circ)}^{W(s)^F} (\tilde{\chi} \otimes \xi) \end{array}$$

is bijective.

The lemmas 2.4.2 and 2.4.3 imply the following :

Theorem 2.4.4. *There is a well-defined bijection*

$$\begin{array}{ccc} \operatorname{Irr} W(s)^F & \longrightarrow & \mathcal{E}(\mathbf{G}^F, (s)) \\ \chi & \longmapsto & \mathbf{R}_\chi(s). \end{array}$$

Remark - If necessary, we will write $\mathbf{R}_\chi^{\mathbf{G}}(s)$ for the irreducible character $\mathbf{R}_\chi(s)$ of $\mathbf{G}^F$. By applying theorem 2.4.4 in the case where $\mathbf{G} = \mathbf{G}(s)$ and $s = 1$, we obtain a bijection

$$\begin{array}{ccc} \operatorname{Irr} W(s)^F & \longrightarrow & \mathcal{E}(\mathbf{G}(s)^F, 1) \\ \chi & \longmapsto & \mathbf{R}_\chi^{\mathbf{G}(s)}(1) \end{array}$$

and it is easy to check that the following diagram is commutative :

(2.4.5)

$$\begin{tikzcd} & \operatorname{Irr} W(s)^F \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \mathcal{E}(\mathbf{G}^F, (s)) \arrow[rr, "\nabla_{\mathbf{G},s}"'] & & \mathcal{E}(\mathbf{G}(s)^F, 1). \end{tikzcd}$$

2.5. Induction from a particular subgroup of G. Let $\mathbf{G}'$ be a subgroup of $\mathbf{G}$ containing $\mathbf{G}^\circ$. It is F-stable because F acts trivially on A. There exists a subgroup A' of A such that

$$\mathbf{G}' = \mathbf{G}^\circ \rtimes A'.$$

If we construct $\mathbf{G}'^*$ in the same way as for $\mathbf{G}$, then $\mathbf{G}'^*$ may be identified with a subgroup of $\mathbf{G}^*$. We can also construct $\mathbf{G}'(s)$ so that it is contained in $\mathbf{G}(s)$ and we denote by $W'(s)$ the Weyl group of $\mathbf{G}'(s)$ relatively to $\mathbf{T}_1^\circ(s)$ so that $W'(s)$ is a subgroup of $W(s)$.

Proposition 2.5.1. *Let χ' be an irreducible character of $W'(s)^F$. Suppose*

$$\operatorname{Ind}_{W'(s)^F}^{W(s)^F} \chi' = \sum_{\chi \in \operatorname{Irr} W(s)^F} n_\chi \, \chi.$$

Then

$$\operatorname{Ind}_{\mathbf{G}'^F}^{\mathbf{G}^F} \mathbf{R}_{\chi'}^{\mathbf{G}'}(s) = \sum_{\chi \in \operatorname{Irr} W(s)^F} n_\chi \, \mathbf{R}_\chi^{\mathbf{G}}(s).$$

3. Lusztig functors

Hypothesis : *Throughout this section, and only in this section, A will be assumed abelian.*

3.1. Notation. Let $\mathbf{L}$ be an F-stable Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Let $A_{\mathbf{L}}$ be the image of $\mathbf{L}$ through the composite morphism

$$\mathbf{L} \longrightarrow \mathbf{G} \longrightarrow \mathbf{G}/\mathbf{G}^\circ \longrightarrow A$$

($A_{\mathbf{L}}$ is a subgroup of A). Because A is abelian, we can use the same argument as in [B], 7.6, to assume that $\mathbf{L}$ contains $A_{\mathbf{L}}$. Let $A_{\mathbf{L}}^*$ be the image of $A_{\mathbf{L}}$ under the anti-isomorphism $A \to A^*$.

Let $\mathbf{L}^{\circ *}$ be an F^*-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^{*\circ}$ which is a dual of $\mathbf{L}^\circ$. We can choose $\mathbf{L}^{\circ *}$ to be $A_{\mathbf{L}}^*$-stable, and we define

$$\mathbf{L}^* = \mathbf{L}^{\circ *} \rtimes A_{\mathbf{L}}^*.$$

Then $\mathbf{L}^*$ is an F^*-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}^*$ and $\mathbf{L}^{*\circ} = \mathbf{L}^{\circ *}$.

3.2. Jordan decomposition and Lusztig functors. Let s be a semisimple element in $\mathbf{L}^{*\circ F^*}$. We may assume that s is nice in $\mathbf{G}^*$. Then the subgroup $\mathbf{L}(s)$ of $\mathbf{L}$ following the construction of paragraph 1.4 can be chosen as a subgroup of $\mathbf{G}(s)$. The linear character of $\mathbf{L}(s)^F$ associated to s as defined in paragraph 1.4 is then the restriction of $\hat{s}$ to $\mathbf{L}(s)^F$. It results from this remark and from the transitivity of Lusztig induction functors (*cf.* [B], proposition 6.3.3) that the following diagram is commutative :

(3.2.1)

$$\begin{tikzcd}
\mathcal{E}(\mathbf{L}^F, (s)_{\mathbf{L}^{*F^*}}) \arrow[r, "\nabla_{\mathbf{L},s}"] \arrow[d, "\varepsilon_{\mathbf{L}^\circ}\varepsilon_{\mathbf{G}^\circ} R_{\mathbf{L}}^{\mathbf{G}}"'] & \mathcal{E}(\mathbf{L}(s)^F, 1) \arrow[d, "\varepsilon_{\mathbf{L}^\circ(s)}\varepsilon_{\mathbf{G}^\circ(s)} R_{\mathbf{L}(s)}^{\mathbf{G}(s)}"] \\
\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}}) \arrow[r, "\nabla_{\mathbf{G},s}"'] & \mathcal{E}(\mathbf{G}(s)^F, 1)
\end{tikzcd}$$

The description of the functor $R_{\mathbf{L}(s)}^{\mathbf{G}(s)}$ in [B], theorem 7.6.1, thus provides a description of the functor $R_{\mathbf{L}}^{\mathbf{G}}$ via the commutative diagram 3.2.1.

4. Shintani descent in the general linear group

In this section, we explain the link between the theory of irreducible characters of $\mathbf{G}^F$ and the theory of Shintani descent for the general linear group. For this purpose, we need to consider a particular case :

Hypothesis and notations : *Throughout this section, we assume that* $r = 1$. *We will denote* $d = d_1$ *and* $n = n_1$ *for simplicity. We also assume that* $\sigma = (1, \dots, d)$ *and that* A *is generated by* σ.

4.1. The group $\mathbf{G}^F$. We denote by $\mathbf{G}_1$ the general linear group $\mathbf{GL}_n$ and we endow it with the split Frobenius endomorphism

$$\begin{array}{rccc} F_0: & \mathbf{G}_1 & \longrightarrow & \mathbf{G}_1 \\ & g & \longmapsto & g^{(q)}. \end{array}$$

We denote by ϕ_0 the automorphism of $\mathbf{G}_1^{F_0^d}$ induced by F_0. Then the map

$$\begin{array}{rccc} \theta: & \mathbf{G}_1^{F_0^d} & \longrightarrow & \mathbf{G}^{\circ F} \\ & g & \longmapsto & (g, F_0(g), \dots, F_0^{d-1}(g)) \end{array}$$

is an isomorphism of groups and the following diagram is commutative :

$$\begin{tikzcd} \mathbf{G}_1^{F_0^d} \arrow[r, "\theta"] \arrow[d, "\phi_0"'] & \mathbf{G}^{\circ F} \arrow[d, "\sigma^{-1}"] \\ \mathbf{G}_1^{F_0^d} \arrow[r, "\theta"'] & \mathbf{G}^{\circ F}. \end{tikzcd}$$

This implies that θ can be extended to an isomorphism denoted by

$$\begin{array}{rccc} \tilde{\theta}: & \mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0 \rangle & \longrightarrow & \mathbf{G}^F \\ & g\phi_0^k & \longmapsto & \theta(g)\sigma^{-k} \end{array}$$

for all $g \in \mathbf{G}_1^{F_0^d}$ and $k \in \mathbb{Z}$.

4.2. Shintani descent. Let $g \in \mathbf{G}_1^{F_0}$. By Lang's theorem, there exists $x \in \mathbf{G}_1$ such that $g = x^{-1}F_0^d(x)$. Then $g' = F_0(x)x^{-1}$ belongs to $\mathbf{G}_1^{F_0^d}$ and the map that sends the conjugacy class of g in $\mathbf{G}_1^{F_0}$ to the ϕ_0-conjugacy class of g' in $\mathbf{G}_1^{F_0^d}$ is well-defined and is bijective. We denote it by

$$N_{F_0/F_0^d} : \mathrm{Cl}(\mathbf{G}_1^{F_0}) \longrightarrow H^1(\phi_0, \mathbf{G}_1^{F_0^d})$$

where $H^1(\phi_0, \mathbf{G}_1^{F_0^d})$ denote the set of ϕ_0-conjugacy classes of $\mathbf{G}_1^{F_0^d}$ and $\mathrm{Cl}(\mathbf{G}_1^{F_0})$ the set of conjugacy classes of $\mathbf{G}_1^{F_0}$. If we denote by $\mathcal{C}(\mathbf{G}_1^{F_0^d}.\phi_0)$ (respectively $\mathcal{C}(\mathbf{G}_1^{F_0})$) the space of class functions on $\mathbf{G}_1^{F_0^d}.\phi_0$ obtained by restrictions from class functions on the group $\mathbf{G}_1^{F_0^d}\langle \phi_0 \rangle$ (respectively $\mathbf{G}_1^{F_0}$), then $N_{F_0^d/F_0}$ induces an isomorphism

$$\mathrm{Sh}_{F_0^d/F_0} : \mathcal{C}(\mathbf{G}_1^{F_0^d}.\phi_0) \longrightarrow \mathcal{C}(\mathbf{G}_1^{F_0})$$

called the **Shintani descent** from F_0^d to F_0.

We recall the following theorem :

Theorem 4.2.1 (Shintani)**.** *Let γ_1 be an irreducible character of $\mathbf{G}_1^{F_0^d}$ stable under ϕ_0. Then there exists an extension $\tilde{\gamma}_1$ of γ_1 to $\mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0 \rangle$ such that $\mathrm{Sh}_{F_0^d/F_0}\tilde{\gamma}_1$ is, up to a sign, an irreducible character of $\mathbf{G}_1^{F_0}$.*

4.3. Shintani descent and characters of $\mathbf{G}^F$. We denote by θ^* and $\tilde{\theta}^*$ the isomorphisms of $\overline{\mathbb{Q}}_\ell$-vector spaces

$$\theta^* : \mathcal{C}(\mathbf{G}^{\circ F}) \longrightarrow \mathcal{C}(\mathbf{G}_1^{F_0^d})$$

and

$$\tilde{\theta}^* : \mathcal{C}(\mathbf{G}^{F}) \longrightarrow \mathcal{C}(\mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0 \rangle)$$

induced by θ and $\tilde{\theta}$ respectively.

Let γ° be an irreducible character of $\mathbf{G}^{\circ F}$ and let $\gamma_1 = \theta^*(\gamma^\circ)$. Then γ_1 is ϕ_0-stable if and only if γ° is σ-stable.

Hypothesis : *From now on, we assume that γ_1 is ϕ_0-stable.*

Let s be a nice semisimple element of $\mathbf{G}^{*\circ F^*}$ such that $\gamma^\circ \in \mathcal{E}(\mathbf{G}^{\circ F}, (s)^\circ)$. Then $A(s) = A$ because γ_1 is ϕ_0-stable. Let χ° be the irreducible character of $W^\circ(s)$ (stable under F) such that $\gamma^\circ = R^\circ_{\chi^\circ}(s)$. Then $A(s, \chi^\circ) = A$.

Theorem 4.3.1. *With the above notations, we have :*

(a) *There exists a unique extension $\tilde{\gamma}_1$ of γ_1 to $\mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0 \rangle$ such that $\mathrm{Sh}_{F_0^d/F_0}\tilde{\gamma}_1$ is an irreducible character of $\mathbf{G}_1^{F_0}$. We call it the* **Shintani extension** *of γ_1.*

(b) *We have $\tilde{\gamma}_1 = \tilde{\theta}^*(\tilde{R}_{\chi^\circ}(s))$.*

(c) *Let e be a divisor of d and let $\tilde{\gamma}_1^{(e)}$ be the Shintani extension of γ_1 to $\mathbf{G}_1^{F_0^d} \rtimes \langle \phi_0^e \rangle$. Then $\tilde{\gamma}_1^{(e)}$ is the restriction of $\tilde{\gamma}_1$.*

Remark - The result stated in (a) of theorem 4.3.1 is slightly stronger than Shintani's one. It was already known for characters of the principal series (F. Digne - J. Michel, *Fonctions $\mathcal{L}$ des variétés de Deligne-Lusztig et descente de Shintani*, Mém. Soc. Math. France (N.S) **20** (1985)).

Proof - By theorem 4.2.1, (a), (b) and (c) are immediate consequence of the following

Lemma 4.3.2. *$\tilde{R}^{\mathbf{G}}_{\chi^\circ}(s)(\sigma^e)$ is a positive integer for all $e \in \mathbb{Z}$.*

Proof of lemma 4.3.2 - Let $e \in \mathbb{Z}$. We first prove that

$$(\star) \qquad \varepsilon_{\mathbf{G}^\circ(s)^{\sigma^e}} = \varepsilon_{\mathbf{G}^\circ(s)} \qquad \text{and} \qquad \varepsilon_{(\mathbf{G}^\circ)^{\sigma^e}} = \varepsilon_{\mathbf{G}^\circ}.$$

Because $\mathbf{G}^\circ(s)$ is a direct product of groups of the same type as $\mathbf{G}^\circ$, it is sufficient to prove the result for $\mathbf{G}^\circ$. But $(\mathbf{T}_0^\circ)^\sigma$ is a maximal split subtorus of $\mathbf{G}^\circ$ so it is a maximal split subtorus of $(\mathbf{G}^\circ)^{\sigma^e}$. That proves $(\star)$.

Let $\tilde{\boldsymbol{\chi}}_e$ be the irreducible character of $W^\circ(s)^{\sigma^e} \rtimes \langle \sigma \rangle$ associated to χ° as in 2.1 (it was denoted $\tilde{\boldsymbol{\chi}}_{\sigma^e}$ but we just want to have simpler notations).

Then, by formulas 2.3.1 and $(\star)$, we have

$$\tilde{R}^{\mathbf{G}}_{\chi^\circ}(s)(\sigma^e) = \frac{\varepsilon_{\mathbf{G}^\circ(s)^{\sigma^e}}\varepsilon_{(\mathbf{G}^\circ)^{\sigma^e}}}{|W^\circ(s)^{\sigma^e}|} \sum_{w \in W^\circ(s)^{\sigma^e}} \tilde{\boldsymbol{\chi}}_e(w\sigma)\ R^{\mathbf{G}^\circ \rtimes \langle \sigma^e \rangle}_{\mathbf{T}_w(s,\sigma^e)}(\hat{s})(\sigma^e).$$

Using [DM2], theorem 4.13, we get

$$\tilde{R}^{\mathbf{G}}_{\chi^\circ}(s)(\sigma^e) = \frac{\varepsilon_{\mathbf{G}^\circ(s)^{\sigma^e}}\varepsilon_{(\mathbf{G}^\circ)^{\sigma^e}}}{|W^\circ(s)^{\sigma^e}|} \sum_{w \in W^\circ(s)^{\sigma^e}} \tilde{\boldsymbol{\chi}}_e(w\sigma)\ \dim R^{(\mathbf{G}^\circ)^{\sigma^e}}_{(\mathbf{T}_w(s,\sigma^e)^\circ)^{\sigma^e}}(1).$$

But this last formula gives the degree of an irreducible character of $((\mathbf{G}^\circ)^{\sigma^e})^F$ (*cf.* [LS], theorem 3.2). ■

References

[B] C. Bonnafé, *Produits en couronne de groupes linéaires*, accepted for publication in the Journal of Algebra.

[DM1] F. Digne - J. Michel, *Representations of finite groups of Lie type*, London Math. Soc. Students Texts **21**, Cambridge University Press (1991).

[DM2] F. Digne - J. Michel, *Groupes réductifs non connexes*, Ann. Scient. Éc. Norm. Sup. 4^{e} série **27** (1994), pp 345-406.

[LS] G. Lusztig - B. Srinivasan, *The characters of the finite unitary groups*, J. Algebra **49** (1977), pp 167-171.

[S] T. Shintani, *Two remarks on irreducible characters of finite general linear groups*, J. Math. Soc. Japan **28** (1976), pp 396-414.