AUTOMORPHISMS OF COXETER GROUPS AND LUSZTIG'S CONJECTURES FOR HECKE ALGEBRAS WITH UNEQUAL PARAMETERS

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Abstract. Let (W,S) be a Coxeter system, let G be a finite solvable group of automorphisms of (W,S) and let φ be a weight function which is invariant under G. Let φ_G denote the weight function on W^G obtained by restriction from φ. The aim of this paper is to compare the $\mathbf{a}$-function, the set of Duflo involutions and the Kazhdan-Lusztig cells associated to (W,φ) and to (W^G,φ_G), provided that Lusztig's Conjectures hold.

Contents

Let (W,S) be a Coxeter system, with S finite, let Γ be a totally ordered abelian group and let $\varphi : W \to \Gamma$ be a *weight function* such that $\varphi(s) > 0$ for all $s \in S$.

Let G be a group of automorphisms of W stabilizing S and φ. We denote by φ_G the restriction of φ to the fixed points subgroup W^G. If $\omega \in S/G$ (the orbit set) is

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such that W_ω ($= \langle \omega \rangle$) is finite, we denote by s_ω the longest element of the standard parabolic subgroup W_ω and we set $S_G = \{s_\omega \mid \omega \in S/G$ and W_ω is finite$\}$. Then it is well-known that (W^G, S_G) is a Coxeter system and that $\varphi_G : W^G \to \Gamma$ is a weight function (such that $\varphi_G(s_\omega) > 0$ for all $\omega \in S/G$).

To the datum (W, S, Γ, φ) are associated a Hecke algebra $\mathcal{H}(W, S, \Gamma, \varphi)$ over the ring $\mathbb{Z}[\Gamma]$, a Kazhdan-Lusztig basis $(C_w)_{w \in W}$ of $\mathcal{H}(W, S, \Gamma, \varphi)$, equivalence relations $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$ and two functions $\mathbf{a} : W \to \Gamma$ and $\Delta : W \to \Gamma$ (see [L]). We set $\mathcal{D} = \{w \in W \mid \mathbf{a}(w) = \Delta(w)\}$. To the datum $(W^G, S_G, \Gamma, \varphi_G)$, we associate similarly $\sim_{\mathcal{L}}^G$, $\sim_{\mathcal{R}}^G$, $\sim_{\mathcal{LR}}^G$, $\mathbf{a}_G$, Δ_G and $\mathcal{D}_G$. The main result of this paper is the following:

Theorem A. *Assume that G is a finite solvable group and that Lusztig's conjectures (P_1), (P_2), (P_3), (P_4) in* [L, Chapter 14] *hold for the datum $(W^H, S_H, \Gamma, \varphi_H)$ for all subgroups H of G. Let x, $y \in W^G$. Then:*

(a) $\mathbf{a}_G(x) = \mathbf{a}(x)$.
(b) $\mathcal{D}_G = \mathcal{D} \cap W^G$.
(c) *Assume moreover that Lusztig's Conjecture (P_{13}) in* [L, Chapter 14] *hold for the datum $(W^H, S_H, \Gamma, \varphi_H)$ for all subgroups H of G. If $? \in \{\mathcal{L}, \mathcal{R}\}$, then $x \sim_? y$ if and only if $x \sim_?^G y$.*
(d) *Assume moreover that Lusztig's Conjectures (P_9) and (P_{13}) in* [L, Chapter 14] *hold for the datum $(W^H, S_H, \Gamma, \varphi_H)$ for all subgroups H of G. Then $x \sim_{\mathcal{LR}} y$ if and only if $x \sim_{\mathcal{LR}}^G y$.*

The proof of this Theorem makes essential use of reduction modulo p. Indeed, an easy induction argument reduces immediately the problem to the case where G is a p-group for some prime number p. The main ingredient is then the following: the natural stupid map $\mathcal{H}(W^G, S_G, \Gamma, \varphi_G) \to \mathcal{H}(W, S, \Gamma, \varphi)^G$ is not a morphism of algebras in general. However, if we denote by $\mathrm{Br}_G(\mathcal{H}(W, S, \Gamma, \varphi))$ the quotient of $\mathcal{H}(W, S, \Gamma, \varphi)^G$ by the two-sided ideal $\sum_{H<G} \mathrm{Tr}_H^G(\mathcal{H}(W, S, \Gamma, \varphi)^H)$ (*Brauer's quotient*, see for instance [T, Page 91]), then:

Proposition B. *Assume that G is a finite p-group. Then the natural linear map $\mathcal{H}(W^G, S_G, \Gamma, \varphi_G) \to \mathrm{Br}_G(\mathcal{H}(W, S, \Gamma, \varphi)^G)$ is a morphism of algebras whose kernel is generated by p. Moreover, it preserves the Kazhdan-Lusztig basis.*

1. The set-up

1.A. The group (W, S). Let (W, S) be a Coxeter system (with S finite), let $\ell : W \to \mathbb{N}$ denote the length function, let Γ be a totally ordered abelian group and let $\varphi : W \to \Gamma$ be a *weight function* [L, §3.1] that is, a map such that $\varphi(ww') = \varphi(w) + \varphi(w')$ whenever $\ell(ww') = \ell(w) + \ell(w')$.

Let A be the group algebra $\mathbb{Z}[\Gamma]$: we will use an exponential notation for A, namely $A = \bigoplus_{\gamma\in\Gamma} \mathbb{Z}e^\gamma$, where $e^\gamma \cdot e^{\gamma'} = e^{\gamma+\gamma'}$ for all γ, $\gamma' \in \Gamma$. If $a = \sum_{\gamma\in\Gamma} a_\gamma e^\gamma \in A$, we denote by $\deg a$ (resp. $\operatorname{val} a$) the *degree* (resp. the *valuation*) of a, that is, the element γ of Γ such that $a_\gamma \neq 0$ and which is maximal (resp. minimal) for this condition (by convention, $\deg 0 = -\infty$ and $\operatorname{val} 0 = +\infty$).

We shall denote by $\mathcal{H}$ the Hecke algebra associated to the datum (W, S, Γ, φ). It is a free A-module, with standard basis $(T_w)_{w\in W}$, and the multiplication is entirely determined by the following rules:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'); \\ (T_s - e^{\varphi(s)})(T_s + e^{-\varphi(s)}) = 0 & \text{if } s \in S. \end{cases}$$

Note that this implies that T_w is invertible in $\mathcal{H}$ for all $w \in W$. This algebra is endowed with an A-anti-linear involution $\bar{\ } : \mathcal{H} \to \mathcal{H}$ which is determined by the following properties:

$$\begin{cases} \overline{e^\gamma} = e^{-\gamma} & \text{if } \gamma \in \Gamma, \\ \overline{T}_w = T_{w^{-1}}^{-1} & \text{if } w \in W. \end{cases}$$

By [L, Theorem 5.2], there exists a unique element $C_w \in \mathcal{H}$ such that

$$\begin{cases} \overline{C}_w = C_w, \\ C_w \equiv T_w \mod \mathcal{H}_{<0}, \end{cases}$$

where $\mathcal{H}_{<0} = \bigoplus_{w\in W} A_{<0} T_w$, and where $A_{<0} = \bigoplus_{\gamma<0} \mathbb{Z}e^\gamma$.

Let $\tau : \mathcal{H} \to A$ be the unique A-linear map such that

$$\tau(T_w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $w \in W$, we set

$$\Delta(w) = -\deg \tau(C_w),$$

and we denote by n_w the coefficient of $e^{-\Delta(w)}$ in $\tau(C_w)$. Finally, if x, $y \in W$, we write

$$C_x C_y = \sum_{z\in W} h_{x,y,z} C_z,$$

where the $h_{x,y,z}$'s are in A and satisfy $\overline{h_{x,y,z}} = h_{x,y,z}$.

1.B. The group (W^G, S_G). Let G be a group of automorphisms of W such that, for all $\sigma \in G$, we have

$$\sigma(S) = S \quad \text{and} \quad \varphi \circ \sigma = \varphi.$$

If I is a subset of S, we denote by W_I the (standard parabolic) subgroup of W generated by I. If $\omega \in S/G$ is such that W_ω is finite, we denote by s_ω the longest element of W_ω. We denote by S_G the set of s_ω, where ω runs over the set of G-orbits in S such that W_ω is finite. Recall the following proposition [H, Corollary 3.5 and Proof of Proposition 3.4]:

Proposition 1.1. *(W^G, S_G) is a Coxeter system. Let $\ell_G : W^G \to \mathbb{N}$ denote the corresponding length function. If x and $y \in W^G$, then $\ell(xy) = \ell(x) + \ell(y)$ if and only if $\ell_G(xy) = \ell_G(x) + \ell_G(y)$.*

Let

$$\begin{array}{rccc} \varphi_G : & W^G & \longrightarrow & \Gamma \\ & w & \longmapsto & \varphi(w) \end{array}$$

denote the restriction of φ to W^G. Then, by Proposition 1.1,

(1.2) *φ_G is a weight function.*

Therefore, we can define $\mathcal{H}_G$, $\mathcal{H}_{G,<0}$, T_w^G, C_w^G, τ_G, Δ_G, n_z^G and $h_{x,y,z}^G$ with respect to $(W^G, S_G, \Gamma, \varphi_G)$ in a similar way as $\mathcal{H}$, $\mathcal{H}_{<0}$, T_w, C_w, τ, Δ, n_z and $h_{x,y,z}$ were defined with respect to (W, S, Γ, φ).

2. Brauer quotient

Hypothesis and notation. *From now on, and until the end of this paper, we fix a prime number p and we assume that G is a finite p-group.*

2.A. Definition. For all the facts contained in this subsection, the reader may refer to [T, §11]: even though this reference deals only with $\mathcal{O}$-algebras (where $\mathcal{O}$ is a commutative complete local noetherian $\mathbb{Z}_p$-algebra) which are $\mathcal{O}$-modules of finite type, the proofs can be applied almost word by word to our more general situation.

Let R be a commutative ring and let M be an RG-module. If H is a subgroup of G, we set

$$\begin{array}{rccc} \mathrm{Tr}_H^G : & M^H & \longrightarrow & M^G \\ & m & \longmapsto & \displaystyle\sum_{\sigma \in [G/H]} \sigma(m). \end{array}$$

We also define

$$\mathrm{Tr}(M) = \sum_{H<G} \mathrm{Tr}_H^G(M^H).$$

This is an R-submodule of M^G, containing pM^G. The *Brauer quotient* $\mathrm{Br}_G(M)$ is then defined by

$$\mathrm{Br}_G(M) = M^G / \mathrm{Tr}(M)$$

and we denote by $\mathrm{br}_G : M^G \to \mathrm{Br}_G(M)$ the canonical map.

Lemma 2.1. *Assume that $pR \neq R$ and that M admits an R-basis $\mathcal{B}$ which is permuted by the action of G. Then $\mathrm{Br}_G(M)$ is a free R/pR-module with basis $(\mathrm{br}_G(b))_{b\in\mathcal{B}^G}$.*

If M is an R-algebra and if G acts on M by automorphisms of algebra, then $\mathrm{Tr}(M)$ is a two-sided ideal of M^G and so $\mathrm{Br}_G(M)$ is an R-algebra. Of course, br_G is a morphism of algebras in this case. We recall the following result:

Lemma 2.2. *Assume that $pR \neq R$, that M is an R-algebra, that G acts on M by automorphisms of algebra, that M admits an R-basis $\mathcal{B}$ which is permuted by G and let us write $ab = \sum_{c\in\mathcal{B}} \lambda_{a,b,c}c$ for a, $b \in \mathcal{B}$. If a, $b \in \mathcal{B}^G$, then*

$$\mathrm{br}_G(a)\,\mathrm{br}_G(b) = \sum_{c\in\mathcal{B}^G} \pi(\lambda_{a,b,c})\,\mathrm{br}_G(c),$$

where $\pi : R \to R/pR$ is the canonical morphism.

2.B. Applications to Hecke algebras. Since G stabilizes S and φ, it also acts on $\mathcal{H}$ by automorphisms of A-algebra (by $\sigma(T_w) = T_{\sigma(w)}$ for all $w \in W$). Moreover, it permutes the standard basis $(T_w)_{w\in W}$, so it follows from Lemma 2.1 that:

Corollary 2.3. $(\mathrm{br}_G(T_w))_{w\in W^G}$ *is an $\mathbb{F}_p[\Gamma]$-basis of the $\mathbb{F}_p[\Gamma]$-algebra $\mathrm{Br}_G(\mathcal{H})$.*

Now, let

$$\mathrm{can}_G : \mathcal{H}_G \longrightarrow \mathrm{Br}_G(\mathcal{H})$$

be the unique A-linear map such that

$$\mathrm{can}_G(T_w^G) = \mathrm{br}_G(T_w)$$

for all $w \in W^G$. The main result of this subsection is the following:

Proposition 2.4. *The map $\mathrm{can}_G : \mathcal{H}_G \longrightarrow \mathrm{Br}_G(\mathcal{H})$ is a surjective morphism of A-algebras whose kernel is $p\mathcal{H}_G$.*

Proof. It follows from Corollary 2.3 that can_G is surjective and that $\mathrm{Ker}(\mathrm{can}_G) = p\mathcal{H}_G$. It remains to show that can_G is a morphism of algebras. First, note that if x, $y \in W^G$ satisfy $\ell_G(xy) = \ell_G(x) + \ell_G(y)$, then $\ell(xy) = \ell(x) + \ell(y)$ (by Proposition 1.1) and so

$$\begin{aligned}\mathrm{can}_G(T_x^G T_y^G) &= \mathrm{can}_G(T_{xy}^G) = \mathrm{br}_G(T_{xy}) \\ &= \mathrm{br}_G(T_x T_y) = \mathrm{br}_G(T_x)\,\mathrm{br}_G(T_y) = \mathrm{can}_G(T_x^G)\,\mathrm{can}_G(T_y^G).\end{aligned}$$

So it remains to show that, if ω is a G-orbit in S such that W_ω is finite, then

$$(?) \qquad \mathrm{br}_G((T_{s_\omega} - e^{\varphi(s_\omega)})(T_{s_\omega} + e^{-\varphi(s_\omega)})) = 0.$$

Since s_ω is the longest element of W_ω, we have [L, Corollary 12.2]

$$C_{s_\omega} = \sum_{w \in W_\omega} e^{-\varphi(w)} T_w$$

and [L, Theorem 6.6 (b)]

$$(T_{s_\omega} - e^{\varphi(s_\omega)}) C_{s_\omega} = 0.$$

But $(W_\omega)^G = \{1, s_\omega\}$. Since $\varphi(w) = \varphi(\sigma(w))$ for all $w \in W_\omega$ and all $\sigma \in G$, we have

$$C_{s_\omega} \equiv T_{s_\omega} + e^{-\varphi(s_\omega)} \mod \mathrm{Tr}(\mathcal{H}).$$

This completes the proof of (?). □

Corollary 2.5. $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{H}_G \simeq \mathrm{Br}_G(\mathcal{H})$.

Corollary 2.6. *If $h \in \mathcal{H}_G$ and $h' \in \mathcal{H}$ are such that $\mathrm{can}_G(h) = \mathrm{br}_G(h')$, then $\tau_G(h) \equiv \tau(h') \mod pA$.*

Proposition 2.7. *If $w \in W^G$, then $\mathrm{can}_G(C_w^G) = \mathrm{br}_G(C_w)$.*

Proof. Let $C = \mathrm{can}_G(C_w^G) - \mathrm{br}_G(C_w)$. Then

$$\overline{C} = C.$$

where $\bar{\ } : \mathrm{Br}_G(\mathcal{H}) \to \mathrm{Br}_G(\mathcal{H})$ is defined by $\overline{\mathrm{br}_G(h)} = \mathrm{br}_G(\overline{h})$ for all $h \in \mathcal{H}^G$. Moreover, there exists a family $(\alpha_w)_{w \in W^G}$ of elements of $\mathbb{F}_p \otimes_{\mathbb{Z}} A_{<0}$ such that

$$C = \sum_{w \in W^G} \alpha_w \, \mathrm{br}_G(T_w).$$

Assume that $C \neq 0$ and let w be maximal (for the Bruhat order) such that $\alpha_w \neq 0$. Then

$$\overline{C} = \overline{\alpha}_w \, \mathrm{br}_G(T_{w^{-1}}^{-1}) + \sum_{\substack{x \in W^G \\ x \neq w}} \overline{\alpha}_x \, \mathrm{br}_G(T_{x^{-1}}^{-1}).$$

Therefore, the coefficient of $\mathrm{br}_G(T_w)$ in $\overline{C}$ is equal to $\overline{\alpha}_w$. But $C = \overline{C}$, so $\alpha_w = \overline{\alpha}_w$. Since $\alpha_w \neq 0$ and $\alpha_w \in \mathbb{F}_p \otimes_{\mathbb{Z}} A_{<0}$, we get a contradiction. So $C = 0$, as desired. □

Corollary 2.8. *If x, y, $z \in W^G$, then $h_{x,y,z} \equiv h_{x,y,z}^G \mod pA$ and $\tau(C_z) \equiv \tau_G(C_z^G) \mod pA$.*

Proof. This follows immediately from Proposition 2.7, from Lemma 2.2 and from Corollary 2.6. □

3. Lusztig's conjectures

3.A. Cells. To (W, S, Γ, φ) are associated preorder relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$ on W as defined in [L, §8.1]. The associated equivalence relations are denoted by $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$ respectively. The equivalence classes for the relation $\sim_{\mathcal{L}}$ (respectively $\sim_{\mathcal{R}}$, respectively $\sim_{\mathcal{LR}}$) are called left (respectively right, respectively two-sided) cells of W (or for (W, S, Γ, φ) if it is necessary to emphasize the weight function).

Similarly, to $(W^G, S_G, \Gamma, \varphi_G)$ are associated preorder relations $\leqslant_{\mathcal{L}}^G$, $\leqslant_{\mathcal{R}}^G$ and $\leqslant_{\mathcal{LR}}^G$ on W. The associated equivalence relations are denoted by $\sim_{\mathcal{L}}^G$, $\sim_{\mathcal{R}}^G$ and $\sim_{\mathcal{LR}}^G$ respectively. We shall compare in this section the (left, right or two-sided) cells of W and the ones of W^G.

3.B. Boundedness. Following Lusztig [L, §13.2], we say that (W, S, Γ, φ) is *bounded* if there exists $\gamma_0 \in \Gamma$ such that $\deg \tau(T_x T_y T_z) \leqslant \gamma_0$ for all x, y and $z \in W$. Lusztig has conjectured [L, Conjecture 13.4] that (W, S, Γ, φ) is always bounded.

> **Hypothesis.** *From now on, and until the end of this paper, we assume that (W, S, Γ, φ) and $(W^G, S_G, \Gamma, \varphi_G)$ are bounded. Recall that p is a prime number and that G is a finite p-group.*

Remark - A finite group is of course bounded. An affine Weyl group is also bounded [L, §13.2]. □

By [L, Lemma 13.5 (b)], this hypothesis allows us to define Lusztig's function $\mathbf{a} : W \to \Gamma$ by

$$\mathbf{a}(z) = \max_{x,y \in W} \bigl(\deg h_{x,y,z}\bigr).$$

If x, y, $z \in W$, we shall denote by $\gamma_{x,y,z^{-1}}$ the unique element of $\mathbb{Z}$ such that

$$h_{x,y,z} \equiv \gamma_{x,y,z^{-1}} e^{\mathbf{a}(z)} \mod \Bigl(\mathop{\oplus}_{\gamma < \mathbf{a}(z)} \mathbb{Z} e^{\gamma}\Bigr).$$

Similarly, we define a function $\mathbf{a}_G : W^G \to \Gamma$ and elements $\gamma_{x,y,z^{-1}}^G$ of $\mathbb{Z}$ (for x, y, $z \in W^G$).

Let $\mathcal{D} = \{z \in W \mid \mathbf{a}(z) = \Delta(z)\}$. If $I \subseteq S$, we denote by $\mathbf{a}_I$ the analogue of the function $\mathbf{a}$ but defined for W_I instead of W: if $z \in W_I$, then

$$\mathbf{a}_I(z) = \max_{x,y \in W_I} \deg h_{x,y,z}.$$

Lusztig's Conjectures for (W,S,Γ,φ). *With the above notation, we have:*

- P_1. *If $z \in W$, then $\mathbf{a}(z) \leqslant \Delta(z)$.*
- P_2. *If $d \in \mathcal{D}$ and if x, $y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.*
- P_3. *If $y \in W$, then there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.*
- P_4. *If $z' \leqslant_{\mathcal{LR}} z$, then $\mathbf{a}(z) \leqslant \mathbf{a}(z')$. Therefore, if $z \sim_{\mathcal{LR}} z'$, then $\mathbf{a}(z) = \mathbf{a}(z')$.*
- P_5. *If $d \in \mathcal{D}$ and $y \in W$ satisfy $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.*
- P_6. *If $d \in \mathcal{D}$, then $d^2 = 1$.*
- P_7. *If x, y, $z \in W$, then $\gamma_{x,y,z} = \gamma_{y,z,x}$.*
- P_8. *If x, y, $z \in W$ satisfy $\gamma_{x,y,z} \neq 0$, then $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$ and $z \sim_{\mathcal{L}} x^{-1}$.*
- P_9. *If $z' \leqslant_{\mathcal{L}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{L}} z$.*
- P_{10}. *If $z' \leqslant_{\mathcal{R}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{R}} z$.*
- P_{11}. *If $z' \leqslant_{\mathcal{LR}} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$, then $z' \sim_{\mathcal{LR}} z$.*
- P_{12}. *If $I \subset S$ and $z \in W_I$, then $\mathbf{a}_I(z) = \mathbf{a}(z)$.*
- P_{13}. *Every left cell $\mathcal{C}$ of W contains a unique element $d \in \mathcal{D}$. If $y \in \mathcal{C}$, then $\gamma_{y^{-1},y,d} \neq 0$.*
- P_{14}. *If $z \in W$, then $z \sim_{\mathcal{LR}} z^{-1}$.*
- P_{15}. *If x, x', y, $w \in W$ are such that $\mathbf{a}(y) = \mathbf{a}(w)$, then*
$$\sum_{y' \in W} h_{w,x',y'} \otimes_{\mathbb{Z}} h_{x,y',y} = \sum_{y' \in W} h_{y',x',y} \otimes_{\mathbb{Z}} h_{x,w,y'}$$
in $A \otimes_{\mathbb{Z}} A$.

Let us recall the following result:

Lemma 3.1. *Assume that Lusztig's Conjectures (P_1), (P_2), (P_3) and (P_4) hold for (W,S,Γ,φ). Then:*

(a) *Lusztig's Conjectures (P_5), (P_6), (P_7) and (P_8) hold for (W,S,Γ,φ).*
(b) *If $d \in \mathcal{D}$, then $\gamma_{d,d,d} = n_d = \pm 1$.*
(c) *If $x \in W$ and if $d \in \mathcal{D}$ is the unique element of W such that $\gamma_{x,x^{-1},d} \neq 0$, then $\gamma_{d,x,x^{-1}} = \pm 1$.*

Proof. (a) is proved in [L, Chapter 14].

(b) By (P_6), we get that $d^2 = 1$. By (P_3), there exists a unique $e \in \mathcal{D}$ such that $\gamma_{d,d,e} \neq 0$. By (P_5), this implies that $\gamma_{d,d,e} = n_e = \pm 1$. By (P_7), this implies that $\gamma_{e,d,d} = \pm 1$. By (P_2), we get that $e = d^{-1} = d$.

(c) If $x \in W$ and if $d \in \mathcal{D}$ is the unique element of W such that $\gamma_{x^{-1},x,d} \neq 0$, then $\gamma_{x,d,x^{-1}} = \gamma_{x^{-1},x,d} = \pm 1$ by (P_7) and (P_5). $\square$

We can now state the main result of this paper:

Theorem 3.2. *Recall that G is a finite p-group. Assume that Lusztig's conjectures (P_1), (P_2), (P_3) and (P_4) hold for both (W, S, Γ, φ) and $(W^G, S_G, \Gamma, \varphi_G)$. Let x and y be two elements of W^G. Then:*

(a) $\mathbf{a}_G(x) = \mathbf{a}(x)$.
(b) $\mathcal{D}_G = \mathcal{D} \cap W^G$ $(= \mathcal{D}^G)$.
(c) *Assume moreover that Lusztig's Conjecture (P_{13}) holds for both (W, S, Γ, φ) and $(W^G, S_G, \Gamma, \varphi_G)$. Then $x \sim_{\mathcal{L}}^G y$ (respectively $x \sim_{\mathcal{R}} y$) if and only if $x \sim_{\mathcal{L}} y$ (respectively $x \sim_{\mathcal{R}} y$).*
(d) *Assume moreover that Lusztig's Conjectures (P_9) and (P_{13}) hold for both (W, S, Γ, φ) and $(W^G, S_G, \Gamma, \varphi_G)$. Then $x \sim_{\mathcal{LR}}^G y$ if and only if $x \sim_{\mathcal{LR}} y$.*

Proof. (a) By Corollary 2.8, we have, for all x, y, $z \in W^G$:

(1) If $\gamma_{x,y,z^{-1}} \not\equiv 0 \mod p$, then $\mathbf{a}(z) \leqslant \mathbf{a}_G(z)$.
(2) If $\gamma^G_{x,y,z^{-1}} \not\equiv 0 \mod p$, then $\mathbf{a}_G(z) \leqslant \mathbf{a}(z)$.

Now let $z \in W^G$. By (P_3), there exists a unique $d \in \mathcal{D}$ such that $\gamma_{z^{-1},z,d} \neq 0$. From the uniqueness, we get that $d \in \mathcal{D}^G \subseteq W^G$. By Lemma 3.1 (c), we get that $\gamma_{z,d,z^{-1}} = \pm 1$. So $\mathbf{a}(z) \leqslant \mathbf{a}_G(z)$ by (1).

The same argument shows that there exists $d \in \mathcal{D}_G$ such that $\gamma^G_{z,d,z^{-1}} = \pm 1$, so (2) can be applied to get that $\mathbf{a}_G(z) \leqslant \mathbf{a}(z)$. The proof of (a) is complete.

Before going further, let us state the following consequence of (a):

Corollary 3.3. *If x, y, $z \in W^G$, then $\gamma_{x,y,z} \equiv \gamma^G_{x,y,z} \mod p$.*

Proof. This follows easily from Theorem 3.2 (a) and Corollary 2.8. $\square$

(b) Let $d \in \mathcal{D}^G$. By Lemma 3.1 (b), we have $n_d = \pm 1$. Moreover, by Corollary 2.8, we have

$$\tau(C_d) \equiv \tau_G(C_d^G) \mod pA.$$

This shows that the coefficient of $e^{\Delta(d)}$ in $\tau_G(C_d^G)$ is non-zero. So $\Delta_G(d) \leqslant \Delta(d)$. But, by (P_1),

$$\mathbf{a}_G(d) \leqslant \Delta_G(d) \leqslant \Delta(d) = \mathbf{a}(d).$$

So $\mathbf{a}_G(d) = \Delta_G(d) = \Delta(d) = \mathbf{a}(d)$ by (a). In particular, $d \in \mathcal{D}_G$.

The same argument shows that, if $d \in \mathcal{D}_G$, then $\Delta(d) \leqslant \Delta_G(d)$ and again we get similarly that $d \in \mathcal{D}$. The proof of (b) is complete.

(c) Let d (respectively e) be the unique element of $\mathcal{D}$ such that $\gamma_{x^{-1},x,d} = \pm 1$ (respectively $\gamma_{y^{-1},y,e} = \pm 1$). By uniqueness, we have d, $e \in \mathcal{D}^G = \mathcal{D}_G$. By Corollary 3.3, we also get $\gamma^G_{x^{-1},x,d} \neq 0$ and $\gamma^G_{y^{-1},y,e} \neq 0$. Therefore, by (P_8), we have

$$x \sim_{\mathcal{L}} d, \quad x \sim_{\mathcal{L}}^G d, \quad y \sim_{\mathcal{L}} e \quad \text{and} \quad y \sim_{\mathcal{L}}^G e.$$

But, by (P_{13}), we have $x \sim_{\mathcal{L}} y$ (respectively $x \sim_{\mathcal{L}}^G y$) if and only if $d = e$. This proves (c).

(d) Recall that (P_9) implies (P_{10}). Moreover, it follows easily from (P_4), (P_9) and (P_{10}) that $\sim_{\mathcal{LR}}$ (respectively $\sim_{\mathcal{LR}}^G$) is the equivalence relation generated by $\sim_{\mathcal{L}}$ and $\sim_{\mathcal{R}}$ (respectively $\sim_{\mathcal{L}}^G$ and $\sim_{\mathcal{R}}^G$). So (d) follows from (c). □

3.C. Asymptotic algebra. Let J (respectively J_G) be the free abelian group with basis $(t_w)_{w \in W}$ (respectively $(t_w^G)_{w \in W}$).

> **Hypothesis.** *In this subsection, and only in this subsection, we assume moreover that Lusztig's Conjectures* (P_1), (P_2),..., (P_{15}) *hold for* (W, S, Γ, φ) *and* $(W^G, S_G, \Gamma, \varphi_G)$.

By [L, §18.3], J (respectively J_G) can be endowed with a structure of associative ring, the multiplication being defined by $t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$ (respectively $t_x^G t_y^G = \sum_{z \in W^G} \gamma^G_{x,y,z^{-1}} t_z^G$). Then it follows immediately from Corollary 3.3 and from Lemma 2.2 that:

Theorem 3.4. *Assume that G is a finite p-group and that Lusztig's Conjectures* (P_1), (P_2),..., (P_{15}) *hold for* (W, S, Γ, φ) *and* $(W^G, S_G, \Gamma, \varphi_G)$. *Then*

$$\mathbb{F}_p \otimes_{\mathbb{Z}} J_G \simeq \mathrm{Br}_G(J).$$

4. OPEN QUESTIONS

The results of this paper should be compared with [L, Chapter 14], where the *quasi-split case* is considered: more particularly, see [L, Lemmas 16.5, 16.6 and 16.14]. This lead to the following questions:

- Does Theorem A hold if G is not solvable? It is probably the case, but a proof should rely on completely different arguments.
- Let $z \in W^G$. Is it true that $\Delta_G(z) \leqslant \Delta(z)$? See [L, Lemma 16.5] for the quasi-split case.
- Let x, $y \in W^G$ be such that $x \leqslant_{\mathcal{L}}^G y$. Is it true that $x \leqslant_{\mathcal{L}} y$? See [L, 16.13 (a)] for the quasi-split case.

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