

AFFINENESS OF DELIGNE-LUSZTIG VARIETIES FOR MINIMAL LENGTH ELEMENTS

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ABSTRACT. We prove that the Deligne-Lusztig varieties associated to elements of the Weyl group which are of minimal length in their twisted class are affine. Our proof differs from the proof of He and Orlik-Rapoport and it is inspired by the case of regular elements, which correspond to the varieties involved in Broué's conjectures.

1. INTRODUCTION

Let p be a prime number, let \mathbb{F} denote an algebraic closure of the finite field with p elements and let \mathbf{G} be a connected reductive algebraic group over \mathbb{F} . We assume that \mathbf{G} is endowed with an isogeny $F : \mathbf{G} \rightarrow \mathbf{G}$ such that F^δ is a Frobenius endomorphism with respect to some \mathbb{F}_q -structure on \mathbf{G} (here, δ is a non-zero natural number, q is a power of p and \mathbb{F}_q denotes the subfield of \mathbb{F} with q elements).

We denote by \mathcal{B} the variety of Borel subgroups of \mathbf{G} and by $\mathcal{B} \times \mathcal{B} = \coprod_{w \in W} \mathcal{O}(w)$ the decomposition into orbits for the diagonal action of \mathbf{G} . Here, W is the Weyl group of \mathbf{G} , with set of simple reflections S corresponding to the orbits of dimension $1 + \dim \mathcal{B}$, and the first and last projections define an isomorphism $\mathcal{O}(w) \times_{\mathcal{B}} \mathcal{O}(w') \xrightarrow{\sim} \mathcal{O}(ww')$ when $\ell(ww') = \ell(w) + \ell(w')$, where $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ is the length function on W associated to S .

Given $w \in W$, we define the *Deligne-Lusztig variety* [4, Definition 1.4] associated to w by

$$\mathbf{X}(w) = \mathbf{X}_{\mathbf{G}}(w) = \{\mathbf{B} \in \mathcal{B} \mid (\mathbf{B}, F(\mathbf{B})) \in \mathcal{O}(w)\}.$$

By studying a class of ample sheaves on $\mathbf{X}(w)$, Deligne and Lusztig proved that these varieties are affine when $q^{1/\delta}$ is larger than the Coxeter number of \mathbf{G} [4, Theorem 9.7].

They proved more generally that the existence of coweights satisfying certain inequalities ensures that $\mathbf{X}(w)$ is affine. Recently, Orlik-Rapoport and He studied this question. Recall that $x, y \in W$ are *F-conjugate* if there exists $a \in W$ such that $y = a^{-1}xF(a)$. By a case-by-case analysis based on Deligne-Lusztig's criterion, they obtained the following result ([12, §5], [9, Theorem 1.3]):

Theorem A (Orlik-Rapoport, He). *If $w \in W$ is an element of minimal length in its F -conjugacy class then $\mathbf{X}(w)$ is affine.*

When w is a Coxeter element, the result is due to Lusztig [11, Corollary 2.8]. The aim of this note is to generalize, and give a more direct proof of Theorem A. As a consequence (and by applying a combinatorial result on elements of minimal length in their F -conjugacy class), we shall obtain a generalisation of Theorem A.

Before stating our results, we need some further notation. We denote by B^+ the *braid monoid* associated to (W, S) . It is the monoid with presentation

$$B^+ = \langle (\underline{x})_{x \in W} \mid \forall x, x' \in W, \ell(xx') = \ell(x) + \ell(x') \Rightarrow \underline{xx'} = \underline{x} \underline{x'} \rangle.$$

The automorphism F of W extends to an automorphism of B^+ still denoted by F .

Given $I \subset S$, let W_I denote the subgroup of W generated by I and let w_I be the longest element of W_I (the element w_S will be denoted by w_0). The main result of this note is the following:

Theorem B. *Let I be an F -stable subset of S and let $w \in W_I$ be such that there exists a positive integer d and $a \in B^+$ with*

$$\underline{w}F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w}_I a.$$

Then $\mathbf{X}(w)$ is affine.

The proof of Theorem B is by a general argument while our deduction of Theorem A relies on combinatorial results on finite Coxeter groups (see [6, Theorem 1.1], [5, §6] and [8, Theorem 7.5]) which are proved by a case-by-case analysis.

There is a case where our criterion can be applied easily. Indeed, if d is a regular number for (W, F) (in the sense of Springer) then by [2, Proposition 6.5], there exists a regular element $w \in W$ such that

$$\underline{w}F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w}_0 \underline{w}_0.$$

Therefore, by Theorem B, the variety $\mathbf{X}(w)$ is affine: this variety is of fundamental interest for the geometric version of Broué's abelian defect group conjecture for finite reductive groups [2, §5.B]. In particular, if $i \neq j$, this conjecture predicts that, as $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$ -modules, the cohomology groups $H_c^i(\mathbf{X}(w), \overline{\mathbb{Q}}_\ell)$ and $H_c^j(\mathbf{X}(w), \overline{\mathbb{Q}}_\ell)$ have no common irreducible constituents.

Finally, note that there exist elements satisfying the criterion of Theorem B but which does not satisfy Deligne-Lusztig's criterion. For instance, if W is of type B_5 (and F acts trivially on W), the element $w = s_1 t s_3 s_2 s_1 t s_1 s_4 s_3 s_2 s_1 t s_1 s_2 s_3$ does not satisfy Deligne-Lusztig's criterion (for $q = 2$) but satisfies $(\underline{w})^5 = (\underline{w}_0)^3$ (here, $S = \{t, s_1, s_2, s_3, s_4\}$, ts_1 has order 4 and $s_i s_{i+1}$ has order 3 for $i = 1, 2, 3$). However, this element w is F -conjugate by cyclic shift (see Section 2 for the definition) to an element which satisfies Deligne-Lusztig's criterion, so the affineness of the variety $\mathbf{X}(w)$ can also be obtained from Deligne-Lusztig's criterion (see Proposition 2).

2. PRELIMINARIES

Levi subgroup. Let us fix an F -stable Borel subgroup \mathbf{B}_0 of \mathbf{G} and an F -stable maximal torus \mathbf{T} of \mathbf{B}_0 . Let \mathbf{U} be the unipotent radical of \mathbf{B}_0 . We identify $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ with W by requiring that $(\mathbf{B}_0, w\mathbf{B}_0w^{-1}) \in \mathcal{O}(w)$.

Let I be an F -stable subset of S , let $\mathbf{P}_I = \mathbf{B}W_I\mathbf{P}$, let \mathbf{V}_I denote the unipotent radical of \mathbf{P}_I and let \mathbf{L}_I denote the unique Levi subgroup of \mathbf{P}_I containing \mathbf{T} . Given $w \in W_I$, there is an isomorphism [10, Lemma 3]

$$\mathbf{X}_{\mathbf{G}}(w) \xrightarrow{\sim} \mathbf{G}^F/\mathbf{V}_I^F \times_{\mathbf{L}_I^F} \mathbf{X}_{\mathbf{L}_I}(w).$$

In particular,

$$(1) \quad \mathbf{X}_{\mathbf{G}}(w) \text{ is affine if and only if } \mathbf{X}_{\mathbf{L}_I}(w) \text{ is affine.}$$

Cyclic shift. If $w, w' \in W$, we say that w and w' are F -conjugate by cyclic shift (and we write $w \xrightarrow{F} w'$) if there exists three sequences $(x_i)_{1 \leq i \leq n}$, $(y_i)_{1 \leq i \leq n}$ and $(w_i)_{1 \leq i \leq n+1}$ of elements of W such that

- (1) $w_1 = w$ and $w_{n+1} = w'$;
- (2) for all $i \in \{1, 2, \dots, n\}$, $w_i = x_i y_i$, $w_{i+1} = y_i F(x_i)$ and $\ell(w_i) = \ell(w_{i+1}) = \ell(x_i) + \ell(y_i)$.

The relation \xrightarrow{F} is an equivalence relation. Two elements which are F -conjugate by cyclic shift have the same length.

Proposition 2. *If $w \xrightarrow{F} w'$, then $\mathbf{X}(w)$ is affine if and only if $\mathbf{X}(w')$ is affine.*

PROOF - By induction, we may assume that there exists x and y in W such that $w = xy$, $w' = yF(x)$ and $\ell(w) = \ell(w') = \ell(x) + \ell(y)$. The result follows from the existence of a purely inseparable morphism $\mathbf{X}(w) \rightarrow \mathbf{X}(w')$ [4, Page 108]. ■

3. PROOF OF THEOREM B

Let I be an F -stable subset of S , let $w \in W_I$ and assume that there exists $a \in B^+$ and a positive integer d such that

$$\underline{w}F(\underline{w}) \cdots F^{d-1}(\underline{w}) = \underline{w}_I a.$$

The aim of this section is to prove that $\mathbf{X}(w)$ is affine (Theorem B). By (1), we may (and we will) assume that $I = S$.

Sequences of elements of W . Given (x_1, \dots, x_r) a sequence of elements of W , we set

$$\mathcal{O}(x_1, \dots, x_r) = \mathcal{O}(x_1) \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathcal{O}(x_r).$$

If (y_1, \dots, y_s) is a sequence of elements of W such that $\underline{x}_1 \cdots \underline{x}_r = \underline{y}_1 \cdots \underline{y}_s$ in B^+ , then $\mathcal{O}(x_1, \dots, x_r) \simeq \mathcal{O}(y_1, \dots, y_s)$ (the varieties are actually canonically isomorphic [3, Application 2]).

Proposition 3. *If there exists $v \in B^+$ such that $\underline{x}_1 \cdots \underline{x}_r = \underline{w}_0 v$, then the variety $\mathcal{O}(x_1, \dots, x_r)$ is affine.*

PROOF - Let $v_1, \dots, v_n \in W$ be such that $\underline{v}_1 \cdots \underline{v}_n = v$. We have $\mathcal{O}(x_1, \dots, x_r) \simeq \mathcal{O}(w_0, v_1, \dots, v_n)$, so it remains to prove that $\mathcal{O}(w_0, v_1, \dots, v_n)$ is affine.

For each $x \in W$, we fix a representative \dot{x} of x in $N_{\mathbf{G}}(\mathbf{T})$. We set

$$\tilde{\mathcal{O}}(x_1, \dots, x_r) = \{(g_0 \mathbf{U}, g_1 \mathbf{U}, \dots, g_r \mathbf{U}) \in (\mathbf{G}/\mathbf{U})^{r+1} \mid \forall 1 \leq i \leq r, g_{i-1}^{-1} g_i \in \mathbf{U} \dot{x}_i \mathbf{U}\}.$$

The group \mathbf{T} acts on the right on $\tilde{\mathcal{O}}(x_1, \dots, x_r)$ as follows:

$$(g_0 \mathbf{U}, g_1 \mathbf{U}, \dots, g_r \mathbf{U}) * t = (g_0 t \mathbf{U}, g_1^{x_1^{-1}} t \mathbf{U}, \dots, g_r^{x_r^{-1} \cdots x_1^{-1}} t \mathbf{U}).$$

The canonical map

$$\begin{aligned} \tilde{\mathcal{O}}(x_1, \dots, x_r) &\longrightarrow \mathcal{O}(x_1, \dots, x_r) \\ (g_0 \mathbf{U}, g_1 \mathbf{U}, \dots, g_r \mathbf{U}) &\longmapsto (g_0 \mathbf{B}_0 g_0^{-1}, g_1 \mathbf{B}_0 g_1^{-1}, \dots, g_r \mathbf{B}_0 g_r^{-1}) \end{aligned}$$

identifies $\mathcal{O}(x_1, \dots, x_r)$ with the quotient of $\tilde{\mathcal{O}}(x_1, \dots, x_r)$ by \mathbf{T} : indeed, both varieties are smooth (hence normal), the above map is smooth (hence separable) and it is easily checked that its fibers are precisely the \mathbf{T} -orbits. Since \mathbf{T} acts freely on $\tilde{\mathcal{O}}(x_1, \dots, x_r)$, and since the quotient of an affine variety by a free action of a torus is affine, [1, Corollary 8.21], the result will follow if we are able to prove that

$$(?) \quad \tilde{\mathcal{O}}(w_0, v_1, \dots, v_n) \text{ is affine.}$$

Let us now prove (?). Since the map $\mathbf{U} \times \mathbf{U} \rightarrow \mathbf{U} \dot{w}_0 \mathbf{U}$, $(x, y) \mapsto x \dot{w}_0 y$ is an isomorphism of varieties, it is easily checked that the map

$$\mathbf{G} \times \mathbf{U} \dot{v}_1 \mathbf{U} \times \cdots \times \mathbf{U} \dot{v}_n \mathbf{U} \longrightarrow \tilde{\mathcal{O}}(w_0, v_1, \dots, v_n)$$

that sends (g, h_1, \dots, h_n) to

$$(g, g \dot{w}_0, g \dot{w}_0 h_1, g \dot{w}_0 h_1 h_2, \dots, g \dot{w}_0 h_1 \dots h_n)$$

is an isomorphism of varieties. This shows the statement (?) and completes the proof of the Proposition. ■

Introducing Frobenius. The morphism

$$\mathbf{X}(w) \rightarrow \mathcal{B}^d, \mathbf{B} \mapsto (\mathbf{B}, F(\mathbf{B}), \dots, F^{d-1}(\mathbf{B}))$$

identifies $\mathbf{X}(w)$ with the closed subvariety $\Delta_d \cap \mathcal{O}(w, F(w), \dots, F^{d-1}(w))$, where $\Delta_d = \{(\mathbf{B}, F(\mathbf{B}), \dots, F^{d-1}(\mathbf{B})) \mid \mathbf{B} \in \mathcal{B}\}$ is a closed subvariety of \mathcal{B}^d . By Proposition 3, the variety $\mathcal{O}(w, F(w), \dots, F^{d-1}(w))$ is affine, hence $\mathbf{X}(w)$ is affine as well. The proof of Theorem B is complete.

4. PROOF OF THEOREM A

Let C be an F -conjugacy class in W and C_{\min} its subset of elements of minimal length. Let d be the smallest positive integer k such that $wF(w)\dots F^{k-1}(w) = 1$ and F^k acts as the identity on W for $w \in C_{\min}$. Following [6, Theorem 1.1] (in the split case) and [5, Definition 5.3] (in the general case), we say that an element $w \in C_{\min}$ is *good* if there exists a sequence $I_1 \supseteq I_2 \supseteq \dots \supseteq I_r$ of subsets of S such that

$$(*) \quad \underline{w}F(\underline{w}) \dots F^{d-1}(\underline{w}) = \underline{w}_{I_1}^2 \underline{w}_{I_2}^2 \dots \underline{w}_{I_r}^2$$

in B^+ .

Proposition 4. *If w is a good element of C_{\min} , then $\mathbf{X}(w)$ is affine.*

PROOF - By Theorem B, it remains to show that the subset I_1 of the identity $(*)$ is F -stable. Let I be the set of simple reflections occurring in a reduced expression of w (note that I does not depend on the choice of the reduced expression [7, Corollary 1.2.3]). Then the set of $s \in S$ such that \underline{s} occurs in a reduced expression of $\underline{w}F(\underline{w}) \dots F^{d-1}(\underline{w})$ is equal to $I \cup F(I) \cup \dots \cup F^{d-1}(I)$ (by looking at the left-hand side of $(*)$) and is also equal to I_1 (by looking at the right-hand side). Since F^d acts as the identity on W , we get that I_1 is F -stable. ■

Let us now come back to the proof of Theorem A. Let $w \in C_{\min}$. Let I be the minimal F -stable subset of S such that $w \in W_I$. By (1), we may assume that $I = S$. Now, if $w' \in C_{\min}$, then $w \xrightarrow{F} w'$ (see [6, Theorem 1.1] for the split case, [5, §6] for twisted exceptional groups and [8, Theorem 7.5] for twisted classical groups), so $\mathbf{X}(w)$ is affine if and only if $\mathbf{X}(w')$ is affine by Proposition 2. Therefore, the result follows from Proposition 4 and the next Theorem:

Theorem 6 (Geck-Michel, Geck-Kim-Pfeiffer, He). *There exists a good element in C_{\min} .*

PROOF - By standard arguments (see [5, §5.5]), we may assume that W is irreducible. If F acts trivially on W , the Theorem is [6, Theorem 1.1]. If F does not act trivially and W is not of type A , this is [5, §5.5]. When W is of type A and F acts non trivially on W , this follows from [8, Corollary 7.25]. ■

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