On domino insertion and Kazhdan–Lusztig cells in type B_n

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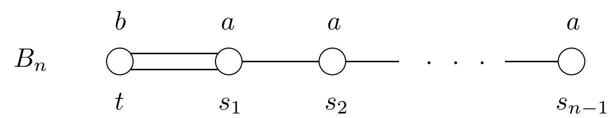
Summary. Based on empirical evidence obtained using the CHEVIE computer algebra system, we present a series of conjectures concerning the combinatorial description of the Kazhdan–Lusztig cells for type B_n with unequal parameters. These conjectures form a far-reaching extension of the results of Bonnafé and Iancu obtained earlier in the so-called "asymptotic case". We give some partial results in support of our conjectures.

1 Introduction and the main conjectures

Let W be a Coxeter group, Γ be a totally ordered abelian group and $L\colon W \to \Gamma_{\geqslant 0}$ be a weight function, in the sense of Lusztig [29, §3.1]. This gives rise to various pre-order relations on W, usually denoted by $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$. Let $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$ be the corresponding equivalence relations. The equivalence classes are called the left, right and two-sided cells of W, respectively. They were first defined by Kazhdan and Lusztig [25] in the case where L is the length function on W (the "equal parameter case"), and by Lusztig [28] in general. They play a fundamental role, for example, in the representation theory of finite or p-adic groups of Lie type; see the survey in [29, Chap. 0].

Our aim is to understand the dependence of the Kazhdan–Lusztig cells on the weight function L. We shall be interested in the case where W is a finite Coxeter group. Then unequal parameters can only arise in type $I_2(m)$ (dihedral), F_4 or B_n. Now types $I_2(m)$ and F_4 can be dealt with by computational methods; see [14]. Thus, as far as finite Coxeter groups are concerned, the real issue is to study type B_n with unequal parameters. And in any case, this is the most important case with respect to applications to finite classical groups (unitary, symplectic, and orthogonal).

The purpose of this paper is to present a series of conjectures which would completely and explicitly determine the Kazhdan–Lusztig cells in type B_n for any positive weight function L. We will also establish some relative results in support of these conjectures. So let now $W = W_n$ be a Coxeter group of type B_n, with generating set $S_n = \{t, s_1, \ldots, s_{n-1}\}$ and Dynkin diagram as given below; the "weights" $a, b \in \Gamma_{>0}$ attached to the generators of W_n uniquely determine a weight function $L = L_{a,b}$ on W_n.



If b is "large" with respect to a, more precisely, if $b > (n-1)a$, then we are in the "asymptotic case" studied in [5] (see also [3, Prop. 5.1 and Cor. 5.2] for the determination of the exact bound). In general, we expect that the combinatorics governing the cells in type B_n are provided by the

"*domino insertion of a signed permutation into a* 2*-core*";

see [26], [27], [32] (see also §3). Having fixed $r \geqslant 0$, let δ_r be the partition with parts $(r, r-1, \ldots, 0)$ (a 2-core). Let $\mathcal{P}_r(n)$ be the set of partitions $\lambda \vdash (\frac{1}{2}r(r+1) + 2n)$ such that λ has 2-core δ_r. Then the domino insertion with respect to δ_r gives a bijection from W_n onto the set of all pairs of standard domino tableaux of the same shape $\lambda \in \mathcal{P}_r(n)$. We write this bijection as $w \to (P^r(w), Q^r(w))$; see [26, §2] for a detailed description.

The following conjectures have been verified for $n \leqslant 6$ by explicit computation using CHEVIE [18] and the program Coxeter developed by du Cloux [8]. For the basic definitions concerning Kazhdan–Lusztig cells, see Lusztig [29].

Conjecture A. *Let $r \geqslant 0$ and assume that $ra < b < (r+1)a$. Then the following hold.*

(a) $w, w' \in W_n$ lie in the same Kazhdan–Lusztig left cell if and only if $Q^r(w) = Q^r(w')$.
(b) $w, w' \in W_n$ lie in the same Kazhdan–Lusztig right cell if and only if $P^r(w) = P^r(w')$.
(c) $w, w' \in W_n$ lie in the same Kazhdan–Lusztig two-sided cell if and only if all of $P^r(w)$, $Q^r(w)$, $P^r(w')$, $Q^r(w')$ have the same shape.

Remark 1.1. If $w \in W_n$, let $\lambda(w) \in \mathcal{P}_r(n)$ denote the shape of $P^r(w)$ (or $Q^r(w)$). Let $\trianglelefteq$ denote the dominance order on partitions. The following refinement of Conjecture A(c) has been checked for $n \leqslant 4$ by using CHEVIE [18]:

$$(\mathrm{c}^+) \qquad w \leqslant_{\mathcal{LR}} w' \quad \text{if and only if} \quad \lambda(w) \trianglelefteq \lambda(w')$$

In general, it is still a conjecture.

Remark 1.2. The 2-core δ_r, the set of partitions $\mathcal{P}_r(n)$, and the parameters $a = 2$, $b = 2r + 1$ (where $\Gamma = \mathbb{Z}$) naturally arise in the representation theory of the finite unitary groups $\mathrm{GU}_N(q)$, where $N = \frac{1}{2}r(r + 1) + 2n$. The Hecke algebra of type B_n with parameters $q^{2r+1}, q^2, \ldots, q^2$ appears as the endomorphism algebra of a certain induced cuspidal representation. The irreducible representations of this endomorphism algebra parametrize the unipotent representations of $\mathrm{GU}_N(q)$ indexed by partitions in $\mathcal{P}_r(n)$; see [6, §13.9]. In this case, Conjecture A(c) is somewhat more precise than [29, Conj. 25.3 (b)] (see §3.3 for more details).

Remark 1.3. If $b > (n - 1)a$ ("asymptotic case"), then domino insertion is equivalent to the generalized Robinson–Schensted correspondence in [5, §3] (see Theorem 3.13). Thus, Conjecture A holds in this case [5, Th. 7.7], [3, Cor. 3.6 and Rem. 3.7]. Also, the refinement (c^+) proposed in Remark 1.1 holds in this case if w and w' have the same t-length [3, Th. 3.5 and Rem. 3.7] (the *t-length* of an element $w \in W_n$ is the number of occurrences of t in a reduced decomposition of w).

Remark 1.4. Since $P^r(w^{-1}) = Q^r(w)$ (see for instance [26, Lemma 7]), we see that the statements (a) and (b) of Conjecture A are equivalent. However, it is not clear that (c) would follow easily from (a) and (b). Indeed, at the time the paper is written, it is conjectured (but not proved in general) that the relation $\sim_{\mathcal{LR}}$ is generated by $\sim_{\mathcal{L}}$ and $\sim_{\mathcal{R}}$. This would follow from Lusztig's Conjectures (P4), (P9), (P10) and (P11).

Remark 1.5. Assume that Conjecture A holds. Then we also conjecture that the Kazhdan–Lusztig basis of the Iwahori–Hecke algebra $\mathcal{H}_n$ associated to W_n and the weight function $L_{a,b}$ is a *cellular basis* in the sense of Graham–Lehrer [22]. See Subsection 2.2 for a more precise statement and applications to the representation theory of non-semisimple specialisations of $\mathcal{H}_n$.

We define the equivalence relation $\simeq_r$ on elements of W_n as follows: we write $w \simeq_r w'$ if and only if $Q^r(w) = Q^r(w')$. An equivalence class for the relation $\simeq_r$ is called a *left r-cell*. In other words, left r-cells are the fibers of the map Q^r. Similarly, we define *right r-cells* as the fibers of the map P^r and *two-sided r-cell* as the fibers of the map $\lambda : W_n \to \mathcal{P}_r(n)$.

Conjecture A(a) says that if $ra < b < (r+1)a$, then the left r-cells coincide with the Kazhdan–Lusztig left cells. Parts (b) and (c) of Conjecture A contain similar statements for the right and two-sided cells. The next conjecture is concerned with the left, right and two-sided Kazhdan–Lusztig cells whenever $b \in \mathbb{N}^* a$.

Conjecture B. *Assume that $b = ra$ for some $r \geqslant 1$. Then the Kazhdan–Lusztig left (resp. right, resp. two-sided) cells of W_n are the smallest subsets of W_n which are at the same time unions of left (resp. right, resp. two-sided) $(r - 1)$-cells and left (resp. right, resp. two-sided) r-cells.*

We will give a combinatorially more precise version of Conjecture B in §4.

Remark 1.6. (a) If $r \geqslant n$ then, since the left r-cells and the left $(r-1)$-cells coincide, then the Conjecture B holds ("asymptotic case", see Remark 1.3).

(b) There is one case which is not covered by Conjectures A and B: it is when $b > ra$ for every $r \in \mathbb{N}$. But this case is exactly the case which is dealt with in [5, Th. 7.7] (and [3, Cor. 3.6] for the determination of two-sided cells) and it leads to the same partition into left and two-sided cells as the case where $(a, b) = (2, 2n-1)$ for instance (see Remark 1.3).

(c) The fundamental difference between the cases where $b \in \{a, 2a, \ldots, (n-1)a\}$ and $b \notin \{a, 2a, \ldots, (n-1)a\}$ is already appearant in [29, Chap. 22], where the "constructible representations" are considered. Conjecturally, these are precisely the representations given by the various left cells of W. By [29, Chap. 22], the constructible representations are all irreducible if and only if $b \notin \{a, 2a, \ldots, (n-1)a\}$.

(d) Again, in Conjecture B, the statement concerning left cells is equivalent to the statement concerning right cells. However, the statement concerning two-sided cells would then follow if one could prove that the relation $\sim_{\mathcal{LR}}$ is generated by the relations $\sim_{\mathcal{L}}$ and $\sim_{\mathcal{R}}$.

(e) Conjectures A and B are consistent with analogous results for type F_4 (see [14] as far as Conjecture A is concerned; Geck also checked that an analogue of Conjecture B holds in type F_4).

In Section 2, we will discuss representation-theoretic issues related to Conjecture A. In Sections 3 and 4, we will present a number of partial results in support of our conjectures.

2 Leading matrix coefficients and cellular bases

Let W be a finite Coxeter group with generating set S. Let Γ be a totally ordered abelian group. Let $L\colon W \to \Gamma$ be a weight function in the sense of Lusztig [29, §3.1]. Thus, we have $L(ww') = L(w) + L(w')$ for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$ where $l\colon W \to \mathbb{N}$ is the usual length function with respect to S (where $\mathbb{N} = \{0, 1, 2, \ldots\}$). Let $A = \mathbb{Z}[\Gamma]$ be the group ring of Γ. It will be denoted exponentially: in other words, $A = \oplus_{\gamma \in \Gamma} \mathbb{Z} v^\gamma$ and $v^\gamma v^{\gamma'} = v^{\gamma+\gamma'}$. If $\gamma_0 \in \Gamma$, let $A_{>\gamma_0} = \oplus_{\gamma > \gamma_0} \mathbb{Z} v^\gamma$. We define similarly $A_{\geqslant \gamma_0}$, $A_{<\gamma_0}$ and $A_{\leqslant \gamma_0}$.

Let $\mathcal{H} = \mathcal{H}_A(W, S, L)$ be the corresponding Iwahori–Hecke algebra. Then $\mathcal{H}$ is free over A with basis $(T_w)_{w \in W}$; the multiplication is given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ T_{sw} + (v^{L(s)} - v^{-L(s)}) T_w & \text{if } l(sw) = l(w) - 1, \end{cases}$$

where $w \in W$ and $s \in S$. For basic properties of W and $\mathcal{H}$, we refer to [21].

2.1 Leading matrix coefficients

We now recall the basic facts concerning the leading matrix coefficients introduced in [12]. First, since Γ is an ordered group, the ring A is integral. Similarly, the group algebra $\mathbb{R}[\Gamma]$ is integral; we denote by $K = \mathbb{R}(\Gamma)$ its field of fractions.

Extending scalars from A to the field K, we obtain a finite dimensional K-algebra $\mathcal{H}_K = K \otimes_A \mathcal{H}$, with basis $(T_w)_{w \in W}$. It is well-known that $\mathcal{H}_K$ is split semisimple and abstractly isomorphic to the group algebra of W over K; see, for example, [19, Remark 3.1]. Let $\mathrm{Irr}(\mathcal{H}_K)$ be the set of irreducible characters of $\mathcal{H}_K$. We write this set in the form

$$\mathrm{Irr}(\mathcal{H}_K) = \{\chi_\lambda \mid \lambda \in \Lambda\},$$

where Λ is some finite indexing set. If $\lambda \in \Lambda$, we denote by d_λ the degree of χ_λ. We have a symmetrizing trace $\tau \colon \mathcal{H}_K \to K$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$; see [21, §8.1]. The fact that $\mathcal{H}_K$ is split semisimple yields that

$$\tau = \sum_{\lambda \in \Lambda} \frac{1}{c_\lambda} \chi_\lambda \qquad \text{where } 0 \neq c_\lambda \in \mathbb{R}[\Gamma].$$

The elements c_λ are called the *Schur elements*. There is a unique $a(\lambda) \in \Gamma_{\geqslant 0}$ and a positive real number r_λ such that

$$c_\lambda \in r_\lambda\, v^{-2a(\lambda)} + A_{>-2a(\lambda)};$$

see [12, Def. 3.3]. The number $a(\lambda)$ is called the a-invariant of χ_λ. Using the *orthogonal representations* defined in [12, §4], we obtain the *leading matrix coefficients* $c_{w,\lambda}^{ij} \in \mathbb{R}$ for $\lambda \in \Lambda$ and $1 \leqslant i,j \leqslant d_\lambda$. See [12, §4] for further general results concerning these coefficients.

Following [19, Def. 3.3], we say that

- $\mathcal{H}$ is *integral* if $c_{w,\lambda}^{ij} \in \mathbb{Z}$ for all $\lambda \in \Lambda$ and $1 \leqslant i,j \leqslant d_\lambda$;
- $\mathcal{H}$ is *normalized* if $r_\lambda = 1$ for all $\lambda \in \Lambda$.

The relevance of these notions is given by the following result.

Theorem 2.1 (See [12, §4] and [19, Lemma 3.8]). *Assume that $\mathcal{H}$ is integral and normalized.*

(a) We have $c_{w,\lambda}^{ij} \in \{0, \pm 1\}$ for all $w \in W$, $\lambda \in \Lambda$ and $1 \leqslant i,j \leqslant d_\lambda$.
(b) For any $\lambda \in \Lambda$ and $1 \leqslant i,j \leqslant d_\lambda$, there exists a unique $w \in W$ such that $c_{w,\lambda}^{ij} \neq 0$; we denote that element by $w = w_\lambda(i,j)$. The correspondence $(\lambda, i, j) \mapsto w_\lambda(i,j)$ defines a bijective map

$$\{(\lambda, i, j)\} \mid \lambda \in \Lambda, 1 \leqslant i,j \leqslant d_\lambda\} \longrightarrow W.$$

(c) For a fixed $\lambda \in \Lambda$ and $1 \leqslant k \leqslant d_\lambda$,
(i) $\mathfrak{L}_{\lambda,k} := \{w_\lambda(i,k) \mid 1 \leqslant i \leqslant d_\lambda\}$ is contained in a left cell;

(ii) $\mathfrak{R}_{\lambda,k} := \{w_\lambda(k,j) \mid 1 \leqslant j \leqslant d_\lambda\}$ *is contained in a right cell.*

We note that the correspondence $(\lambda, i, j) \mapsto w_\lambda(i,j)$ may be regarded as a representation-theoretic generalization of the classical Robinson–Schensted correspondence for the symmetric group.

Remark 2.2. Assume that Lusztig's conjectures **(P1)**–**(P15)** in [29, §14.2] hold for $\mathcal{H}$. Assume also that $\mathcal{H}$ is normalized and integral. Combining [15, Corollary 4.8] and [19, Lemma 3.10], we conclude that the sets $\mathfrak{L}_{\lambda,k}$ and $\mathfrak{R}_{\lambda,k}$ are precisely the left cells and the right cells of W, respectively.

Now let $W = W_n$ be the Coxeter group of type B_n as in Section 1; let $\mathcal{H}_n$ be the associated Iwahori–Hecke algebra with respect to the weight function $L = L_{a,b}$ where $a, b \geqslant 0$.

Proposition 2.3. *Assume that* $a > 0$ *and* $b \notin \{a, 2a, \ldots, (n-1)a\}$. *Then* $\mathcal{H}_n$ *is integral and normalized.*

Proof. The fact that $\mathcal{H}_n$ is normalized follows from the explicit description of $a(\lambda)$ in [29, Prop. 22.14]. To show that $\mathcal{H}_n$ is integral we follow once more the discussion in [19, Example 3.6] where we showed that $\mathcal{H}_n$ is integral if $b > (n-1)a$. So we may, and we will, assume from now on that $b < (n-1)a$. Since $b \notin \{a, 2a, \ldots, (n-1)a\}$, there exists a unique $r \geqslant 0$ such that $ra < b < (r+1)a$. Given $\lambda \in \Lambda$, let $\tilde{S}^\lambda$ be the Specht module constructed by Dipper–James–Murphy [7]. There is a non-degenerate $\mathcal{H}_n$-invariant bilinear form $\langle\ ,\ \rangle_\lambda$ on $\tilde{S}^\lambda$. Let $\{f_t \mid t \in \mathbb{T}_\lambda\}$ be the orthogonal basis constructed in [7, Theorem 8.11], where $\mathbb{T}_\lambda$ is the set of all standard bitableaux of shape λ. Using the recursion formula in [9, Prop. 3.8], it is straightforward to show that, for each basis element f_t, there exist integers $s_t, a_{ti}, b_{tj}, c_{tk}, d_{tl} \in \mathbb{Z}$ such that $a_{ti} \geqslant 0$, $b_{tj} \geqslant 0$, and

$$\langle f_t, f_t\rangle_\lambda = v^{2s_t a} \cdot \frac{\prod_i (1 + v^{2a} + \cdots + v^{2a_{ti}a})}{\prod_j (1 + v^{2a} + \cdots + v^{2b_{tj}a})} \cdot \frac{\prod_k \big(1 + v^{2(b+c_{tk}a)}\big)}{\prod_l \big(1 + v^{2(b+d_{tl}a)}\big)}.$$

In [19, Example 3.6], we noticed that we also have $b + c_{tk}a > 0$ and $b + d_{tl}a > 0$ if $b > (n-1)a$, and this allowed us to deduce that $\mathcal{H}_n$ is integral in that case. Now, if we only assume that $ra < b < (r+1)a$, then $b + d_{tl}a$ and $b + d_{tl}a$ will no longer be strictly positive, but at least we know that they cannot be zero. Thus, there exist $h_t, h_t', m_{tk}, m_{tl}' \in \mathbb{Z}$ such that

$$\prod_k \big(1 + v^{2(b+c_{tk}a)}\big) = v^{2h_t} \prod_k \big(1 + v^{2m_{tk}}\big) \qquad \text{where } m_{tk} > 0,$$
$$\prod_l \big(1 + v^{2(b+d_{tl}a)}\big) = v^{2h_t'} \prod_l \big(1 + v^{2m_{tl}'}\big) \qquad \text{where } m_{tl}' > 0.$$

Hence, setting

$$\tilde{f}_t := v^{-s_t a - h_t + h'_t} \cdot \Big(\prod_j (1 + v^{2a} + \cdots + v^{2b_{tj}a})\Big) \cdot \Big(\prod_l (1 + v^{2m'_{tl}})\Big) \cdot f_t,$$

we obtain $\langle \tilde{f}_t, \tilde{f}_t\rangle_\lambda \in 1 + v\mathbb{Z}[v]$ for all t. We can then proceed exactly as in [19, Example 3.6] to conclude that $\mathcal{H}_n$ is integral. □

The above result, in combination with Theorem 2.1, provides a first approximation to the left and right cells of W_n. By Remark 2.2, the sets $\mathcal{L}_{\lambda,k}$ and $\mathcal{R}_{\lambda,k}$ should be precisely the left and right cells, respectively. In this context, Conjecture A would give an explicit combinatorial description of the correspondence $(\lambda, i, j) \mapsto w_\lambda(i, j)$.

2.2 Cellular bases

Let us assume that we are in the setting of Conjecture A. As announced in Remark 1.5, we believe that then the Kazhdan–Lusztig basis of $\mathcal{H}_n$ will be cellular in the sense of Graham–Lehrer [22]. To state this more precisely, we have to introduce some further notation. Let $(C_w)_{w\in W}$ be the Kazhdan–Lusztig basis of $\mathcal{H}_n$; the element C_w is uniquely determined by the conditions that

$$\overline{C}_w = C_w \qquad \text{and} \qquad C_w \equiv T_w \mod \mathcal{H}_{n,>0},$$

where $\mathcal{H}_{n,>0} = \sum_{w\in W_n} A_{>0} T_w$ and the bar denotes the ring involution defined in [29, Lemma 4.2]. Furthermore, let $* : \mathcal{H}_n \to \mathcal{H}_n$ be the unique anti-automorphism such that $T_w^* = T_{w^{-1}}$ for all $w \in W_n$. We also have $C_w^* = C_{w^{-1}}$ for any $w \in W_n$.

Now assume that $a > 0$ and $b \notin \{a, 2a, \dots, (n-1)a\}$. If $b < (n-1)a$, let $r \geqslant 0$ be such that $ra < b < (r+1)a$. If $b > (n-1)a$, let r be any natural number greater than or equal to $n-1$.

We set $\Lambda_r := \mathcal{P}_r(n)$ and consider the partial order on Λ_r given by the dominance order $\trianglelefteq$ on partitions. For $\lambda \in \Lambda_r$, let $M_r(\lambda)$ denote the set of standard domino tableaux of shape λ. If $(S,T) \in M_r(\lambda) \times M_r(\lambda)$, let $C_r(S,T) := C_w$ where $(S,T) = (P^r(w), Q^r(w))$.

Conjecture C. *With the above notation, $(\Lambda_r, M_r, C_r, *)$ is a cell datum in the sense of Graham–Lehrer [22, Def. 1.1].*

The existence of a cellular structure has strong representation-theoretic applications. For the remainder of this section, assume that Conjecture C is true. Let $\theta\colon A \to k$ be a ring homomorphism into a field k. Extending scalars from A to k, we obtain a k-algebra $\mathcal{H}_{n,k} := k \otimes_A \mathcal{H}_n$ which will no longer be semisimple in general. The theory of cellular algebras [22] provides, for every $\lambda \in \Lambda_r$, a *cell module* S^λ of $\mathcal{H}_{n,k}$, endowed with an $\mathcal{H}_{n,k}$-equivariant bilinear form ϕ^λ. We set

$$D^\lambda := S^\lambda / \operatorname{rad} \phi^\lambda \qquad \text{for every } \lambda \in \Lambda_r.$$

Let $\Lambda_r^\circ := \{D^\lambda \mid \lambda \in \Lambda_r \text{ such that } \phi^\lambda \neq 0\}$. Then we have

$$\operatorname{Irr}(\mathcal{H}_{n,k}) = \{D^\lambda \mid \lambda \in \Lambda_r^\circ\}; \qquad \text{see Graham–Lehrer [22, Thm 3.4].}$$

Thus, we obtain a natural parametrization of the irreducible representations of $\mathcal{H}_{n,k}$ by the set $\Lambda_r^\circ \subseteq \Lambda_r$.

Remark 2.4. Assume that $b > (n-1)a > 0$. Then Conjecture C holds by [16, Cor. 6.4]. In this case, the set Λ_r° is determined explicitly by Dipper–James–Murphy [7] and Ariki [1]. Finally, Iancu–Pallikaros [24] show that the cell modules S^λ are canonically isomorphic to the Specht modules defined by Dipper–James–Murphy [7].

Remark 2.5. Now consider arbitrary values of a, b such that $a > 0$ and $b \notin \{a, 2a, \ldots, (n-1)a\}$. Then, assuming that the conjectured relation (c^+) in Remark 1.1 holds, a description of the set Λ_r follows from the results of Geck–Jacon [20] on *canonical basic sets*. Indeed, one readily shows that the set Λ_r° coincides with the *canonical basic set* determined by [20]. Thus, by the results of [20], we have explicit combinatorial descriptions of Λ_r° in all cases. Note that these descriptions heavily depend on a, b and $\theta\colon A \to k$.

It is shown in [17] that, if $a = 2$ and $b = 1$ or 3, then the sets Λ_r° parametrize the modular principal series representations of the finite unitary groups.

3 Domino insertion

The aim of this section is to describe the domino insertion algorithm and to provide some theoretical evidences for Conjecture A. For this purpose we will see W_n as the group of permutations w of $\{-1, -2, \ldots, -n\} \cup \{1, 2, \ldots, n\}$ such that $w(-i) = -w(i)$ for any i. The identification is as follows: t corresponds to the transposition $(1, -1)$ and s_i to $(i, i+1)(-i, -i-1)$. If $r \leqslant n$, we identify W_r with the subgroup of W_n generated by $S_r = \{t, s_1, s_2, \ldots, s_{r-1}\}$. The symmetric group of degree n will be denoted by $\mathfrak{S}_n$: when necessary, we shall identify it in the natural way with the subgroup of W_n generated by $\{s_1, s_2, \ldots, s_{n-1}\}$. Let $t_1 = t$ and, if $1 \leqslant i \leqslant n-1$, let $t_{i+1} = s_i t_i s_i$. As a signed permutation, t_i is just the transposition $(i, -i)$.

Remark 3.1. Since we shall be interested in various descent sets of elements of W_n, we state here for our future needs the following two easy facts. Let $w \in W_n$. Then the following hold.

(a) If $1 \leqslant i \leqslant n-1$, then $\ell(ws_i) > \ell(w)$ if and only if $w(i) < w(i+1)$.
(b) If $1 \leqslant i \leqslant n$, then $\ell(wt_i) > \ell(w)$ if and only if $w(i) > 0$.

3.1 Partitions and Tableaux

We refer to [26, 32] for further details of the material in this section. We shall assume some familiarity with (standard) Young tableaux.

Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{l(\lambda)} > 0)$ be a partition of $n = |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)}$. We will not distinguish between a partition λ and its *Young diagram* (often denoted $D(\lambda)$). Our Young diagrams will be drawn in the English notation so that the boxes are upper-left justified. When λ and μ are partitions satisfying $\mu \subset \lambda$ we will use λ/μ to denote the shape corresponding to the set-difference of the diagrams of λ and μ. We call λ/μ a *domino* if it consists of exactly two squares sharing an edge.

The 2-*core* (or just core) $\widetilde{\lambda}$ of a shape λ is obtained by removing dominoes from λ, keeping the shape a partition, until this is no longer possible. The partition $\widetilde{\lambda}$ does not depend on how these dominoes are removed. Every 2-core has the shape of a staircase $\delta_r = (r, r-1, \ldots, 0)$ for some integer $r \geqslant 0$.

We denote the set of partitions by $\mathcal{P}$ and the set of partitions with 2-core δ_r by $\mathcal{P}_r$. The set of all partitions λ satisfying the conditions:

$$\widetilde{\lambda} = \delta_r \text{ and } |\lambda| = |\delta_r| + 2n$$

will be denoted $\mathcal{P}_r(n)$. Note that $\mathcal{P} = \cup_{r,n}\mathcal{P}_r(n)$ is a disjoint union.

A *(standard) domino tableau* D of *shape* $\lambda \in \mathcal{P}_r(n)$ consists of a tiling of the shape $\lambda/\widetilde{\lambda}$ by dominoes and a filling of the dominoes with the integers $\{1, 2, \ldots, n\}$, each used exactly once, so that the numbers are increasing when read along either the rows or columns. The *value* of a domino is the number written inside it. We will denote by dom_i the domino with the value i inside. We will also write $\mathrm{sh}(D) = \lambda$ for the shape of D. An equivalent description of the domino tableau D is as the sequence of partitions $\{\widetilde{\lambda} = \lambda^0 \subset \lambda^1 \subset \ldots \subset \lambda^n = \lambda\}$, where $\mathrm{sh}(\mathrm{dom}_i) = \lambda^i/\lambda^{i-1}$. If the values of the dominoes in a tableau D are not restricted to the set $\{1, 2, \ldots, n\}$ (but each value occurs at most once), we will call D an *injective domino tableau*.

We now describe a number of operations on standard Young and domino tableaux needed in the sequel. One may obtain a standard Young tableau $T = T(D)$ from a domino tableau D by replacing a domino with the value i in D by two boxes containing $\bar{i}$ and i in T. The boxes are placed so that T is standard with respect to the order $\bar{1} < 1 < \bar{2} < 2 < \cdots$. If D has shape λ then $T(D)$ will have shape $\lambda/\widetilde{\lambda}$. Suppose now that Y is a standard Young tableau of shape $\widetilde{\lambda}$ filled with letters smaller than any of the letters occurring in D. Define $T_Y(D)$ by "filling" in the empty squares in $T(D)$ with the tableau Y.

Let T be a standard Young tableau and i a letter occurring in T. The *conversion* process proceeds as follows (see [23, 32]). Replace the letter i in T with another letter j. The resulting tableau may not be standard, so we repeatedly swap j with its neighbours until the tableau is standard. We say that the value i has been converted to j.

Now let T be any standard Young tableau filled with barred $\bar{i}$ and non-barred letters i. Define T^{neg} by successively converting barred letters $\bar{i}$ to negative letters $-i$, starting with the smallest letters. The main fact that we shall need is that the operation "neg" is invertible. We refer the reader to [32] for a full discussion of these operations.

3.2 The Barbasch-Vogan domino insertion algorithm

The Robinson-Schensted correspondence establishes a bijection

$$\pi \leftrightarrow (P(\pi), Q(\pi))$$

between permutations $\pi \in \mathfrak{S}_n$ and pairs of standard tableaux with the same shape and size n (see [33]). Domino insertion generalizes this by replacing the symmetric group with the hyperoctahedral group. It depends on the choice of a core δ_r, and establishes a bijection between W_n and pairs (P^r, Q^r) of standard domino tableaux of the same shape $\lambda \in \mathcal{P}_r(n)$. There are in fact many such bijections but we will be concerned only with the algorithm introduced by Barbasch and Vogan [2] and later given a different description by Garfinkle [10]. We now describe this algorithm following the more modern expositions [26, 32].

Let D be an injective domino tableau with shape λ such that $i > 0$ is a value which does not occur in D. We describe the insertion $E = D \leftarrow i$ (or $E = D \leftarrow -i$) of a horizontal (vertical) domino with value i into D. Let $D_{<i} \subset D$ denote the sub-domino tableau of D containing all dominoes with values less than i. If λ has a 2-core $\lambda = \tilde{\lambda}$, then we will always assume that $\tilde{\lambda} \subset \mathrm{sh}(D_{<i})$. Let $E_{\leqslant i}$ be the domino tableau obtained from $D_{<i}$ by adding an additional vertical domino in the first column or an additional horizontal domino in the first row labeled i.

For $j > i$ we define $E_{\leqslant j}$, supposing that $E_{\leqslant j-1}$ is known. If D contains no domino domino labeled j then $E_{\leqslant j} = E_{\leqslant j-1}$; otherwise let dom_j denote the domino in D labeled j. Let $\mu = \mathrm{sh}(E_{\leqslant j-1})$. We now distinguish four cases:

1. If $\mu \cap \mathrm{dom}_j = \emptyset$ do not touch, then we set $E_{\leqslant j} = E_{\leqslant j-1} \cup \mathrm{dom}_j$.
2. If $\mu \cap \mathrm{dom}_j = (k, l)$ is exactly one square in the k-th row and l-th column, then we add a domino containing j to $E_{\leqslant j-1}$ to obtain the tableau $E_{\leqslant j}$ which has shape $\mu \cup \mathrm{dom}_j \cup (k+1, l+1)$.
3. If $\mu \cap \mathrm{dom}_j = \mathrm{dom}_j$ and dom_j is horizontal, then we bump the domino dom_j to the next row, by setting $E_{\leqslant j}$ to be the union of $E_{\leqslant j-1}$ with an additional (horizontal) domino with value j one row below that of dom_j.
4. If $\mu \cap \mathrm{dom}_j = \mathrm{dom}_j$ and dom_j is vertical, then we bump the domino dom_j to the next column, by setting $E_{\leqslant j}$ to be the union of $E_{\leqslant j-1}$ with an additional (vertical) domino with value j one column to the right of dom_j.

Finally we let $E = \lim_{j \to \infty} E_{\leqslant j}$.

Let $w = w(1)w(2)\cdots w(n) \in W_n$ be a hyperoctahedral permutation written in one-line notation. Thus, for each i, we have $w(i) \in \{\pm1, \pm2, \ldots, \pm n\}$; furthermore, $|w(1)||w(2)|\cdots|w(n)| \in \mathfrak{S}_n$ is a usual permutation. Let δ_r be a 2-core assumed to be fixed. Then the insertion tableau $P^r(w)$ is defined as $((\ldots((\delta_r \leftarrow w(1)) \leftarrow w(2))\cdots) \leftarrow w(n))$. The sequence of shapes obtained in the process defines another standard domino tableau called the recording tableau $Q^r(w)$ of $w \in W_n$. The insertion tableau $P^r(w)$ can of course be defined for any sequence $w = w(1)w(2)\cdots w(n)$ such that $|w(i)| \neq |w(j)|$ for $i \neq j$.

The following theorem is due to Barbasch-Vogan [2] and Garfinkle [10] when $r = 0, 1$ and extended by van Leeuwen [27] to larger cores.

Theorem 3.2. *Fix $r \geqslant 0$. The domino insertion algorithm defines a bijection between $w \in W_n$ and pairs (P, Q) of standard domino tableaux of the same shape lying in $\mathcal{P}_r(n)$. This bijection satisfies the equality $P^r(w) = Q^r(w^{-1})$.*

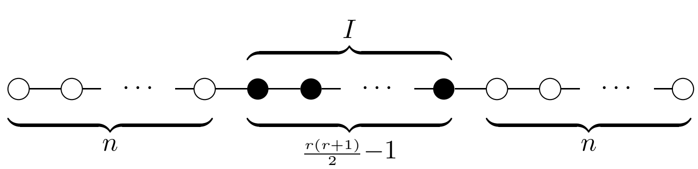
It is easy to see that the bijectivity in Theorem 3.2 together with Conjecture A would imply that the relevant left cell representations are irreducible. This is consistent with the conjecture that left cell representations for W_n are irreducible for "generic parameters" and in particular if $b \notin \{a, 2a, \ldots, (n-1)a\}$ (see Proposition 2.3).

We have computational evidences for Conjectures A and B: it has been checked for $n \leqslant 6$ by using CHEVIE [18] and Coxeter [8]. In the rest of this section, we shall give theoretical evidences for Conjecture A (induction of cells, multiplication by the longest element, link to [29, Conj. 25.3], asymptotic case, quasi-split case, right descent sets, coplactic relations).

3.3 Conjecture A and Lusztig's Conjecture 25.3

There is an alternative description (in the case where $r = 0$, 1, it is in fact the original description of Barbasch and Vogan) of domino insertion. As we now explained, it is related to [29, Conj. 25.3]. Let us fix in this subsection a Coxeter group (W, S) of type $A_{2n+r(r+1)/2-1}$. Let σ be the unique non-trivial automorphism of W such that $\sigma(S) = S$. If J is a subset of S, we denote by W_J the parabolic subgroup of W generated by J and let w_J denote the longest element of W_J.

Let I be the unique connected (when we view it as a subdiagram of the Dynkin diagram of (W, S)) subset of S of cardinality $r(r+1)/2 - 1$ (or 0 if $r = 0$) such that $\sigma(I) = I$:



Let $\mathcal{W}$ denote the subgroup of W consisting of all elements w such that $wW_Iw^{-1} = W_I$ and w has minimal length in wW_I (see [29, §25.1]). If Ω is a σ-orbit in $S \setminus I$, we set $s_\Omega = w_{I\cup\Omega}w_I$. If $0 \leqslant i \leqslant n-1$, let Ω_i denote the orbit of σ in $S \setminus I$ consisting of elements which are separated from I by i nodes in the Dynkin diagram. Then $\{\Omega_0, \Omega_1, \ldots, \Omega_{n-1}\}$ is the set of orbits of σ in $S \setminus I$. Moreover, there is a unique morphism of groups $\iota_r : W_n \to \mathcal{W}^\sigma$ that sends t to s_{Ω_0} and s_i to s_{Ω_i} (for $1 \leqslant i \leqslant n-1$). It is an isomorphism of groups (see [29, §25.1]).

The morphism ι_r can be described explicitly in the language of signed permutations. First identify W with the permutation group of the following $2n + r(r-1)/2$ elements (ordered according to the ordering of S):

$$\{-n < -(n-1) < \cdots < -1 < 0_1 < 0_2 < \cdots < 0_{r(r-1)/2} < 1 < 2 < \cdots n\}$$

so that the subgroup W_I (which is isomorphic to $\mathfrak{S}_{r(r+1)/2}$) acts on the elements $\{0_1, 0_2, \ldots, 0_{r(r-1)/2}\}$. Let $w = w(1)w(2)\cdots w(n) \in W_n$. Then the two-line notation of $\iota_r(w)$ is given by

$$\text{(1)} \qquad \begin{pmatrix} -n & \cdots & -1 & 0_1 & \cdots & 0_{r(r-1)/2} & 1 & 2 & \cdots & n \\ -w(n) & \cdots & -w(1) & 0_1 & \cdots & 0_{r(r-1)/2} & w(1) & w(2) & \cdots & w(n) \end{pmatrix}.$$

Now, let $\mathbf{c}_0$ denote the two-sided cell of W_I which has "shape" δ_r. If w, $w' \in \mathfrak{S}_n$, we write $w \simeq_{\mathfrak{S}} w'$ if $Q(w) = Q(w')$ (the equivalence relation $\simeq_{\mathfrak{S}}$ defines the Robinson-Schensted left cells of $\mathfrak{S}_n$, which coincide with the Kazhdan-Lusztig left cells [25, §5]).

Theorem 3.3. *Fix $x \in \mathbf{c}_0$. Let $w \in W_n$, $r \geqslant 0$ and $\pi = \iota_r(w)x \in \mathfrak{S}_{2n+r(r+1)/2}$. Then we have*

$$(T_{P(x)}(P^r(w)))^{\mathrm{neg}} = P(\pi) \qquad \text{and} \qquad (T_{Q(x)}(Q^r(w)))^{\mathrm{neg}} = Q(\pi).$$

Since neg *is invertible, in particular $w \simeq_r w'$ if and only if $\iota_r(w)x \simeq_{\mathfrak{S}} \iota_r(w')x$.*

For the construction of $T_{P(x)}(P^r(w))$ in Theorem 3.3 we are using the ordering $0_1 < 0_2 < \cdots < 0_{r(r-1)/2} < \bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{n} < n$. In the case where $r = 0$ or 1, Theorem 3.3 is essentially [27, Theorem 4.2.3] with different notation. To generalize the result to all $r \geqslant 0$ we follow the approach of [32].

Proof. For the case $r = 0$ the theorem is exactly [32, Theorem 32]. We now explain, assuming familiarity with [32], how to extend the result to larger cores. It is shown in [32, Lemma 31] that a domino insertion $D \leftarrow i$ can be imitated by *doubly mixed insertion*, denoted P_{m^*}. The proof of [32, Lemma 31] is local, and remains valid when we replace $D_{<i}$ by any Young tableau of the same shape, filled with "small" letters. More precisely, their proof shows that $T_{P(x)}(P^r(w))$ can be obtained by doubly mixed insertion of a "biword" w^{dup} (explicitly defined in [32]) into $P(x)$. Thus one has

$$T_{P(x)}(P^r(w)) = P_{m^*}(x \sqcup w^{\mathrm{dup}}), \tag{2}$$

where $a \sqcup b$ denotes the word obtained from concatenating a and b. In the notation of [32], x here is a biword with no bars so that $P_{m^*}(x) = P(x)$.

Now [32, Theorem 21 and Proposition 14] connect doubly mixed insertion with usual Schensted insertion via the equation

$$P_{m^*}(u)^{\mathrm{neg}} = P(u^{\mathrm{inv\,neg\,inv\,neg}}). \tag{3}$$

The operation denoted "inv neg inv neg" in [32] applied to $x \sqcup w^{\mathrm{dup}}$ coincides with our inclusion $i_r(w)x$. Combining (2) and (3) one obtains

$$(T_{P(x)}(P^r(w)))^{\mathrm{neg}} = P(\pi).$$

The statement about recording tableau is obtained analogously, or by using the equation $Q(\pi) = P(\pi^{-1}) = P(x^{-1}i_r(w)^{-1}) = P(i_r(w^{-1})x^{-1})$. $\square$

Remark 3.4. Note that the last statement of Theorem 3.3 does not depend on the choice of $x \in \mathbf{c}_0$.

Corollary 3.5. *If $r \geqslant 0$ and if $(a,b) = (2, 2r+1)$, then Conjecture A(c) agrees with [29, Conj. 25.3] for the case (W,S,I,σ) described above.*

3.4 Longest element

Let w_0 denote the longest element of W_n: it is equal to $t_1t_2\dots t_n$ (or to -1 $-2\dots -n$ in the one line notation). It is a classical result that two elements x and y in W_n satisfy $x \sim_{\mathcal{L}} y$ if and only if $w_0x \sim_{\mathcal{L}} w_0y$. The next result shows that the relations $\simeq_r$ share the same property.

Proposition 3.6. *Let $r \geqslant 0$ and let x, $y \in W_n$. Then $x \simeq_r y$ if and only if $w_0x \simeq_r w_0y$.*

Proof. This follows from the easy fact that $P^r(w_0x)$ (resp. $Q^r(w_0x)$) is the conjugate (that is, the transpose with respect to the diagonal) of $P^r(x)$ (resp. $Q^r(x)$), and similarly for y. $\square$

3.5 Induction of cells

Let $m \leqslant n$. Let X_m^n denote the set of elements $w \in W_n$ which have minimal length in wW_m. It is a cross-section of W_n/W_m. By Remark 3.1, an element $x \in W_n$ belongs to X_m^n if and only if $0 < x(1) < x(2) < \cdots < x(m)$. A Theorem of Geck [13] asserts that, if C is a Kazhdan-Lusztig left cell of W_m (associated with the restriction of $L_{a,b}$ to W_m), then $X_m^n C$ is a union of Kazhdan-Lusztig left cells of W_n. The next result show that the same hold if we replace *Kazhdan Lusztig left cell* by *left r-cell*.

Proposition 3.7. *Let $r \geqslant 0$. If C is a left r-cell of W_m, then $X_m^n C$ is a union of left r-cells of W_n.*

Proof. Let w, $w' \in W_n$ and x, $x' \in X_m^n$ be such that $xw \simeq_r x'w'$ (in W_n). We must show that $w \simeq_r w'$ (in W_m). For the purpose of this proof, we shall denote by $(P_n^r(w), Q_n^r(w))$ (resp. $(P_m^r(w), Q_m^r(w))$) the pair of standard domino tableaux obtained by viewing w as an element of W_n (resp. of W_m). Then, since x is increasing on $\{1, 2, \ldots, m\}$ and takes only positive values, the dominoes filled with $\{1, 2, \ldots, m\}$ in the recording tableau $Q_n^r(xw)$ are the same as the one in the recording tableau $Q_n^r(w)$. In particular, $Q_m^r(w)$ is obtained from $Q_n^r(xw)$ by removing the dominoes filled by $\{m+1, m+2, \ldots, n\}$. Similarly, $Q_m^r(w')$ is obtained from $Q_n^r(x'w')$ by removing the dominoes filled by $\{m+1, m+2, \ldots, n\}$. Since $Q_n^r(xw) = Q_n^r(x'w')$ by hypothesis, we have that $Q_m^r(w) = Q_m^r(w')$. In other words, $w \simeq_r w'$ in W_m. □

Corollary 3.8. *Let $r \geqslant 0$ and let x and y be two elements of W_m. Then $x \simeq_r y$ in W_m if and only if $x \simeq_r y$ in W_n.*

The previous corollary shows that it is not necessary to make the ambient group precise when one studies the equivalence relation $\simeq_r$.

Geck's result [13] is valid for any Coxeter group and any parabolic subgroup. We shall investigate now the analogue of Proposition 3.7 for the parabolic subgroup $\mathfrak{S}_n$ of W_n. We denote by $X(n)$ the set of elements $w \in W_n$ which have minimal length in $w\mathfrak{S}_n$. It is a cross-section of $W_n/\mathfrak{S}_n$. By Remark 3.1, an element $w \in W_n$ belongs to $X(n)$ if and only if $w(1) < w(2) < \cdots < w(n)$.

Proposition 3.9. *Let $r \geqslant 0$ and let C be a Robinson-Schensted left cell of $\mathfrak{S}_n$. Then $X(n)C$ is a union of domino left cells for $\simeq_r$.*

Proof. Let $w, w' \in \mathfrak{S}_n$ and $x, x' \in X(n)$ be such that $xw \simeq_r x'w'$ in W_n. We must show that $w \simeq_{\mathfrak{S}} w'$. It is well known that for two words $a_1 a_2 \cdots a_k$ and $b_1 b_2 \cdots b_k$ one has $Q(a_1 a_2 \cdots a_k) = Q(b_1 b_2 \cdots b_k) \implies Q(a_j a_{j+1} \cdots a_l) = Q(b_j b_{j+1} \cdots b_l)$ for any $1 \leqslant j \leqslant l \leqslant k$. Indeed this is Geck's result [13] for $\mathfrak{S}_n$.

By Theorem 3.3, we have $Q(i_r(xw)c) = Q(i_r(x'w')c)$ for any $c \in \mathbf{c}_0$. Treating $i_r(xw)c$ as a word using (1), we thus have

$$Q(xw(1)\; xw(2)\; \cdots\; xw(n)) = Q(x'w'(1)\; x'w'(2)\; \cdots\; x'w'(n)).$$

But $x(1) < x(2) < \cdots < x(n)$ so that this is equivalent to $Q(w) = Q(w')$. □

If $x \in X_m^n$, and if w, $w' \in W_m$ are such that $w \sim_{\mathcal{L}} w'$ then a result of Lusztig [29, Prop. 9.13] asserts that $wx^{-1} \sim_{\mathcal{L}} w'x^{-1}$. The next result shows that the same statement holds if we replace $\sim_{\mathcal{L}}$ by $\simeq_r$.

Proposition 3.10. *Let $r \geqslant 0$, $x \in X_m^n$ and w, $w' \in W_m$ be such that $w \simeq_r w'$. Then $wx^{-1} \simeq_r w'x^{-1}$.*

Proof. Let us use here the notation of the proof of Proposition 3.7. So we assume that $P^r_m(w^{-1}) = P^r_m(w'^{-1})$ and we must show that $P^r_n(xw^{-1}) = P^r_n(xw'^{-1})$. Let $D = ((\dots((\delta_r \leftarrow xw^{-1}(1)) \leftarrow xw^{-1}(2))\cdots) \leftarrow xw^{-1}(m))$ and $D' = ((\dots((\delta_r \leftarrow xw'^{-1}(1)) \leftarrow xw'^{-1}(2))\cdots) \leftarrow xw'^{-1}(m))$. Since $w^{-1}(i) = i$ if $i \geqslant m+1$, we have $P^r_n(xw^{-1}) = ((\dots((D \leftarrow x(m+1)) \leftarrow x(m+2))\cdots) \leftarrow x(n))$ and similarly for $P^r_n(xw'^{-1})$. Therefore, we only need to show that $D = D'$. But, since w^{-1} stabilizes $\{1,2,\dots,m\}$ and since x is increasing on $\{1,2,\dots,m\}$ and takes only positive values, it follows from the domino insertion algorithm that D is obtained from $P^r_m(w^{-1})$ by applying x, or in other words replacing the domino dom_i by $\mathrm{dom}_{x(i)}$. Similarly, D' is obtained from $P^r_m(w'^{-1})$ by applying x. Since $P^r_m(w^{-1}) = P^r_m(w'^{-1})$ by hypothesis, we get that $D = D'$, as desired. □

As for Geck's result, Lusztig's result [29, Prop. 9.13] is valid for any parabolic subgroup of any Coxeter group. The next result is the analogue of Proposition 3.10 for the parabolic subgroup $\mathfrak{S}_n$ of W_n.

Proposition 3.11. *Let $r \geqslant 0$, $x \in X(n)$ and w, $w' \in \mathfrak{S}_n$ be such that $w \simeq_{\mathfrak{S}} w'$. Then $wx^{-1} \simeq_r w'x^{-1}$.*

Proof. We must show that $P^r(xw^{-1}) = P^r(xw'^{-1})$ knowing that $P(w^{-1}) = P(w'^{-1})$. For $u \in W_n$, denote by $u_{\mathfrak{S}}$ the word $u(1)\ u(2)\ \cdots\ u(n)$ and $u_{-\mathfrak{S}}$ the word $-u(n)\ -u(n-1)\ \cdots\ -u(1)$. The equation $P(w^{-1}) = P(w'^{-1})$ gives

$$(4) \qquad P((xw')_{\mathfrak{S}}) = P((xw'^{-1})_{\mathfrak{S}}) \text{ and } P((xw')_{-\mathfrak{S}}) = P((xw'^{-1})_{-\mathfrak{S}}),$$

where we are comparing pairs of standard Young tableaux filled with the set of letters $\{x(1), x(2), \dots, x(n)\}$ (resp. $(\{-x(1), -x(2), \dots, -x(n)\})$.

By Theorem 3.3, it is enough to show that $P(\iota_r(xw^{-1})c) = P(\iota_r(xw'^{-1})c)$ for some fixed $c \in \mathbf{c}_0$. Using (1), we may write $i_r(xw^{-1})c$ in one-line notation as the concatenation $(xw^{-1})_{-\mathfrak{S}} \sqcup c \sqcup (xw^{-1})_{\mathfrak{S}}$. It is well known that if a, a', b are words such that $P(a) = P(a')$ then one has $P(a \sqcup b) = P(a' \sqcup b)$ (this is Lusztig's result [29, Prop. 9.13] for the symmetric group). Combining this with (4) we obtain $P((xw^{-1})_{-\mathfrak{S}} \sqcup c \sqcup (xw^{-1})_{\mathfrak{S}}) = P((xw'^{-1})_{-\mathfrak{S}} \sqcup c \sqcup (xw'^{-1})_{\mathfrak{S}})$, as desired. □

Remark 3.12. The Propositions 3.7, 3.9, 3.10 and 3.11 generalize [4, Prop. 4.8] (which corresponds to the asymptotic case).

3.6 Asymptotic case, quasi-split case

We now prove Conjecture A for $r = 0, 1$ and $r \geqslant n-1$.

Theorem 3.13. *Conjecture A is true for $r \geqslant n-1$.*

Proof. Let $r \geqslant n-1$, and let D be a domino tableau with shape $\lambda \in \mathcal{P}_r(n)$. The dominoes $\{\mathrm{dom}_i \mid i \in \{1, 2, \ldots, n\}\}$ can be decomposed into the two disjoint collections $\mathcal{D}_+ = \{\mathrm{dom}_i \mid \mathrm{dom}_i \text{ is horizontal}\}$ and $\mathcal{D}_- = \{\mathrm{dom}_i \mid \mathrm{dom}_i \text{ is vertical}\}$ such that all the dominoes in $\mathcal{D}_+$ lie strictly above and to the right of all the dominoes in $\mathcal{D}_-$. We call a tableau satisfying this property *segregrated*. If the collection of dominoes $\mathcal{D}_+$ is left justified, and each domino replaced by a single box, one obtains a usual Young tableau. Similarly, if the dominoes $\mathcal{D}_-$ are justified upwards and changed into boxes, one obtains a usual Young tableau.

In other words, D can be thought of as a union of two usual tableau D_+ and D_- so that the union of the values in D_+ and in D_- is the set $\{1, 2, \ldots, n\}$. To be consistent with the remaining discussion we in fact define D_- to be the *conjugate* (reflection in the main diagonal) of the Young tableau obtained from the dominoes $\mathcal{D}_-$, as described above.

Domino insertion is compatible with this decomposition so that the following diagram commutes:

$$\begin{array}{ccc} w & \xrightarrow{\text{domino insertion}} & (P^r(w), Q^r(w)) \\ \big\downarrow & & \big\downarrow \\ [w_+, w_-] & \xrightarrow{\text{RS insertion}} & [P^r_+(w), Q^r_+(w)), (P^r_-(w), Q^r_-(w))] \end{array}$$

Here w_+ denotes the subword of w consisting of positive letters and w_- denotes the subword consisting of negative letters, with the minus signs removed. In [5], it is shown for $r \geqslant n-1$ and $w, w' \in W_n$ that $w \sim_{\mathcal{L}} w'$ if and only if $Q(w_+) = Q(w'_+)$ and $Q(w_-) = Q(w'_-)$; similar results hold for $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$. Since $Q(w_+) = Q^r_+(w)$ and $Q(w_-) = Q^r_-(w)$, we have $w \sim_{\mathcal{L}} w'$ if and only if $Q^r(w) = Q^r(w')$, establishing Conjecture A in this case for left cells. A similar argument works for right cells and two-sided cells, using also the classification of two-sided cells in [3]. □

Theorem 3.14. *Conjecture A is true if $a = 2b$ or if $3a = 2b$.*

Proof. In [28], Lusztig determined the left cells of W_n with parameters $b = (2r+1)a/2$ for $r \in \{0, 1\}$ as follows. When $r \in \{0, 1\}$ we have $I = \emptyset$ in the notation of Section 3.3. The equal parameter weight function L on $\mathfrak{S}_{2n+r}$ restricts to the weight function $L_{b,a}$ on $\iota_r(W_n)$, where $b = (2r+1)a/2$. Lusztig [28, Theorem 11] shows that each left cell of W_n is the intersection of a left cell of $\mathfrak{S}_{2n+r}$ with $\iota_r(W_n)$. Thus $w \simeq_r w'$ in W_n if and only if $\iota_r(w) \simeq_{\mathfrak{S}} \iota_r(w')$ in $\mathfrak{S}_{2n+r}$. When $r \in \{0, 1\}$ there is no need for the element $x \in \mathbf{c}_0$ in Theorem 3.3, so one obtains Conjecture A for $r \in \{0, 1\}$. □

3.7 Right descent sets

If $r \geqslant 0$, let $S_n^{(r)} = \{s_1, s_2, \ldots, s_{n-1}\} \cup \{t_1, \ldots, t_r\}$ (if $r \geqslant n$, then $S_n^{(r)} = S_n^{(n)}$). If $w \in W_n$, let

$$\mathcal{R}_n^{(r)}(w) = \{s \in S_n^{(r)} \mid \ell(ws) < \ell(w)\}$$

be the *extended right descent set* of w. The following proposition is easy:

Proposition 3.15. *Let x and y be two elements of W_n. Then:*

(a) If $x \simeq_r y$, then $\mathcal{R}_n^{(r+1)}(x) = \mathcal{R}_n^{(r+1)}(y)$.
(b) If $b > ra$ and if $x \sim_{\mathcal{L}} y$, then $\mathcal{R}_n^{(r+1)}(x) = \mathcal{R}_n^{(r+1)}(y)$.

Proof. If $r \geqslant n-1$, then statements (a) and (b) are equivalent by Theorem 3.13. But, in this case, (b) has been proved in [4, Prop. 4.5]. So let us assume from now on that $r < n-1$. We shall prove (a) and (b) together. Let us set

$$\mathcal{R}_s(x) = \{s \in \{s_1, \dots, s_{n-1}\} \mid \ell(ws) > \ell(w)\},$$
$$\mathcal{R}_t^{(r)}(x) = \{s \in \{t_1, \dots, t_r\} \mid \ell(ws) > \ell(w)\}.$$

Then $\mathcal{R}_n^{(r)}(x) = \mathcal{R}_s(s) \cup \mathcal{R}_t^{(r)}(x)$.

Write $x = ux'$ and $y = vy'$, with u, $v \in X_{r+1}^n$ and x', $y' \in W_{r+1}$. Since $\ell(ux') = \ell(u) + \ell(x')$ (and similarly for $ux's$ for any $s \in W_r$), we have that $\mathcal{R}_t^{(r)}(x) = \mathcal{R}_t^{(r)}(x')$. Similarly, $\mathcal{R}_t^{(r)}(y) = \mathcal{R}_t^{(r)}(y')$. But, if x and y satisfy (a) or (b), then $\mathcal{R}_t^{(r)}(x') = \mathcal{R}_t^{(r)}(y')$: indeed, this follows from the fact that (a) and (b) have been proved in the asymptotic case and, in case (a), from Proposition 3.7 and, in case (b), from [13].

Now it remains to show that $\mathcal{R}_s(x) = \mathcal{R}_s(y)$ if x and y satisfy (a) or (b). In case (b), this follows from [29, Lemma 8.6]. So assume now that $x \simeq_r y$. Write $x = u'\sigma$ and $y = v'\tau$, with u, $v \in X(n)$ and σ, $\tau \in \mathfrak{S}_n$. As in the previous case, we have $\mathcal{R}_s(x) = \mathcal{R}_s(\sigma)$ and $\mathcal{R}_s(y) = \mathcal{R}_s(\tau)$. Moreover, by Proposition 3.9, we have $\sigma \simeq_{\mathfrak{S}} \tau$. It is well-known that it implies that $\mathcal{R}_s(\sigma) = \mathcal{R}_s(\tau)$. □

Remark 3.16. Proposition 3.15 (a) can also be deduced from [32, Lemma 33] or [26, Lemma 9].

3.8 Coplactic relations

If x and y are two elements of W_n such that $\ell(x) \leqslant \ell(y)$, then we write $x \smile_r y$ if there exists $s \in S_n^{(0)}$ and $s' \in S_n^{(r)}$ such that $y = sx$ and $\ell(s'x) < \ell(x) < \ell(y) < \ell(s'y)$. If $\ell(x) \geqslant \ell(y)$, then we write $x \smile_r y$ if $y \smile_r x$. Let $\equiv_r$ denote the reflexive and transitive closure of $\smile_r$.

Remark 3.17. (a) If $x \equiv_r y$, then $\ell_t(x) = \ell_t(y)$.

(b) If $r' \geqslant r$ and if $x \equiv_r y$, then $x \equiv_{r'} y$ (indeed, if $x \smile_{r'} y$, then $x \smile_r y$: this just follows from the fact that $S_n^{(r)} \subset S_n^{(r')}$). Moreover, the relations $\equiv_n$ and $\equiv_r$ are equal if $r \geqslant n$.

(c) If $r \geqslant n-1$, then $x \equiv_r y$ if and only if $x \equiv_{n-1} y$. Let us prove this statement. By (b) above, we only need to show that, if $x \smile_n y$, then $x \smile_{n-1} y$. For this, we may assume that $\ell(y) > \ell(x)$. So there exists $i \in \{1, 2, \dots, n-1\}$

and $s' \in S_n^{(n)}$ such that $y = s_i x$ and $\ell(s'x) < \ell(x) < \ell(s_i x) < \ell(s' s_i x)$. If $s' \in S_n^{(n-1)}$ then we are done. So we may assume that $s' = t_n$. Therefore, the first inequality says that $x^{-1}(n) < 0$ and the last inequality says that $x^{-1}s_i(n) < 0$. This implies that $i = n-1$. Consequently, we have $x^{-1}(n) < 0$ and $x^{-1}(n-1) > 0$. But, the middle inequality says that $x^{-1}(n-1) < x^{-1}(n)$, so we obtain a contradiction with the fact that $s' = t_n$.

(d) If $b > ra$ and if $r \geqslant n-1$, then it follows from Theorem 3.13, from [5, Prop. 3.8] and from (a), (b) and (c) above that the relations $\sim_{\mathcal{L}}$, $\simeq_r$ and $\equiv_r$ coincide.

Proposition 3.18. *Let x and y be two elements of W_n such that $x \equiv_r y$. Then the following hold:*

(i) $x \simeq_r y$.
(ii) If $b > ra$, then $x \sim_{\mathcal{L}} y$.

Proof. We may, and we will, assume that $x \smile_r y$. By symmetry, we may also assume that $\ell(y) > \ell(x)$. We shall prove (i) and (ii) together. There exists $s \in S_n^{(0)}$ and $s' \in S_n^{(r)}$ such that $y = sx$ and $\ell(s'x) < \ell(x) < \ell(y) < \ell(s'y)$. Two cases may occur:

- If $s' \in S_n^{(0)}$, then write $x = x'u^{-1}$ and $y = y'v^{-1}$ with $x', y' \in \mathfrak{S}_n$ and $u, v \in X(n)$. Then $y' = sx'$, $u = v$ and $\ell(s'x') < \ell(x') < \ell(y') < \ell(s'y')$. It is well-known that it implies that $Q(x') = Q(y')$ (Knuth relations), so $x' \simeq_{\mathfrak{Q}} y'$ and $x' \sim_{\mathcal{L}} y'$. Therefore, since moreover $u = v$, it follows from Proposition 3.11 (resp. [29, Prop. 9.13]) that $x \simeq_r y$ (resp. $x \sim_{\mathcal{L}} y$).

- If $s' \notin S_n^{(0)}$ then we write $s = s_i$ and $s' = t_j$. Then the relations $y = sx$ and $\ell(s'x') < \ell(x') < \ell(y') < \ell(s'y')$ implies that $x'^{-1}(j) < 0$ and $x'^{-1}s_i(j) > 0$. In particular, s and s' belong to W_{r+1}. Now, write $x = x'u^{-1}$ and $y = y'v^{-1}$ with $x', y' \in W_{r+1}$ and $u, v \in X_{r+1}^n$. Then $y' = sx'$, $u = v$ and $\ell(s'x') < \ell(x') < \ell(y') < \ell(s'y')$. By Remark 3.17 (d), this implies that $x' \simeq_r y'$ and, if $b > ra$, that $x' \sim_{\mathcal{L}} y'$. Therefore, since moreover $u = v$, it follows from Proposition 3.11 (resp. [29, Prop. 9.13]) that $x \simeq_r y$ (resp. $x \sim_{\mathcal{L}} y$). □

Even if we have both $\ell_t(x) = \ell_t(y)$ and $x \simeq_r y$ we do not necessarily have $x \equiv_r y$. For example, let $r = 0$, $n = 6$ and take $x = 5\ 6\ 1\ 4\ 2\ {-3}$ and $y = 5\ 6\ {-1}\ 4\ 3\ 2$.

4 Cycles and Conjecture B

4.1 Open and closed cycles

We now describe a more refined combinatorial structure of domino tableaux introduced by Garfinkle [10]. We will mostly follow the setup of [30].

Let D be a domino tableau with shape $\lambda \in \mathcal{P}_r(n)$. We call a square $(i,j) \in D$ *variable* if $i+j$ and r have the same parity, otherwise we call it *fixed*. If the domino dom_i contains the square (k,l) we write $D(k,l) = i$.

Now let (k,l) be the fixed square of dom_i. Suppose that dom_i occupies the squares $\{(k,l),(k+1,l)\}$ or $\{(k,l-1),(k,l)\}$. We define a new domino dom'_i by letting it occupy the squares

1. $\{(k,l),(k-1,l)\}$ if $i < D(i-1,j+1)$,
2. $\{(k,l),(k,l+1)\}$ if $i > D(i-1,j+1)$.

Otherwise dom_i occupies the squares $\{(k,l),(k,l+1)\}$ or $\{(k-1,l),(k,l)\}$. We define a new domino dom'_i by letting it occupy the squares

1. $\{(k,l),(k,l-1)\}$ if $i < D(i+1,j-1)$,
2. $\{(k,l),(k+1,l)\}$ if $i > D(i+1,j-1)$.

Now define the *cycle* $c = c(D,i)$ of D through i to be the smallest union c of dominoes satisfying that (i) $\mathrm{dom}_i \in c$ and (ii) $\mathrm{dom}_j \in c$ if $\mathrm{dom}_j \cap \mathrm{dom}'_j \neq \emptyset$. If c is a cycle of D we let $M(D,c)$ be the domino tableau obtained from D by replacing each domino $\mathrm{dom}_i \in c$ by dom'_i. We call this procedure *moving through* c.

Theorem 4.1 ([10]). *Let D be a domino tableau and c a cycle of D. Then $M(D,c)$ is a standard domino tableau. Furthermore, if C is a set of cycles of D then the tableau $M(D,C)$ obtained by moving through each $c \in C$ is defined unambiguously.*

We call a cycle c *closed* if $M(D,c)$ has the same shape as D; otherwise we call c *open*. Note that each (non-trivial) cycle c is in one of two positions, so that moving through is an invertible operation.

4.2 Evidence for Conjecture B

The notion of open and closed cycles allows us to state a combinatorially more precise version of Conjecture B.

Conjecture D. *Assume that $b = ra$ for some $r \geqslant 1$. Then the following hold for any w, $w' \in W_n$:*

(a) $w \sim_{\mathcal{L}} w'$ *if and only if* $Q^{r-1}(w) = M(Q^{r-1}(w'), C)$ *for a set C of* ***open*** *cycles.*

(b) $w \sim_{\mathcal{R}} w'$ *if and only if* $P^{r-1}(w) = M(P^{r-1}(w'), C)$ *for a set C of* ***open*** *cycles.*

(c) $w \sim_{\mathcal{LR}} w'$ *if and only if some tableau with shape equal to* $\mathrm{sh}(P^{r-1}(w)) = \mathrm{sh}(Q^{r-1}(w))$ *can be obtained from a tableau with shape* $\mathrm{sh}(P^{r-1}(w')) = \mathrm{sh}(Q^{r-1}(w'))$ *by moving through a set of* ***open*** *cycles.*

Remark 4.2. Each cycle c of a domino tableau D is in one of two positions, and by Theorem 4.1 they can be moved independently. Thus Conjecture D would imply that every left cell for the parameters $b = ra$ would be a union of 2^d left cells for the parameters $b = \frac{(2r-1)a}{2}$. Here d is equal to the number open cycles, which do not change the shape of the core, in one of the Q-tableau in Conjecture D. This is consistent with the fact that, for $b = ra$ with $r \geqslant 1$, then the number of irreducible components of a constructible representations is a power of 2 (see [29, Chap. 22]; see also with [28, (12.1)] for the equal parameter case).

We have the following theorem of Pietraho, obtained via a careful study of the combinatorics of cycles.

Theorem 4.3 (Pietraho [31]). *Conjecture B and Conjecture D are equivalent.*

Some special cases of Conjectures B and D are known. The case $b = a$ or $r = 1$ is known as the equal parameter case and is closely connected with the classification of primitive ideals of classical Lie algebras.

Theorem 4.4 (Garfinkle [11]). *Conjecture D is true for $r = 1$.*

The asymptotic case follows from [5, 3].

Theorem 4.5. *Conjecture D holds for $r \geqslant n$.*

Proof. Let D be a domino tableau with shape $\lambda \in \mathcal{P}_q(n)$ such that $q \geqslant n-1$. Then moving through any cycle of D changes the shape of the core of D. Thus (for left cells) the condition $Q^q(w) = M(Q^q(w'), C)$ in Conjecture D is the same as the condition $Q^q(w) = Q^q(w')$. This agrees with the classification given in [5]. A similar argument works for right cells and two-sided cells, also using the classification in [3]. □

Acknowledgements. T. L. was partially supported by NSF DMS–0600677. Parts of the work presented here were done while all four authors enjoyed the hospitality of the Bernoulli center at the EPFL Lausanne (Switzerland), in the framework of the research program "Group representation theory" from January to June 2005.

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