Actions of relative Weyl groups I

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Abstract. We construct a new isomorphism between the endomorphism algebra of an induced cuspidal character sheaf and the group algebra of the relative Weyl group involved. We show that it differs from the isomorphism of Lusztig by a linear character, and we relate this linear character to some stabilizers. Some consequences for characteristic functions of character sheaves are obtained. In the forthcoming second part, we will compute explicitly this linear character whenever the cuspidal local system is supported by the regular unipotent class and, as an application of these methods, we obtain a refinement of the theorem of Digne, Lehrer and Michel on Lusztig restriction of Gel'fand–Graev characters.

General introduction

A theorem of Digne, Lehrer and Michel states that the Lusztig restriction of a Gel'fand–Graev character of a finite reductive group G^F is still a Gel'fand–Graev character [7, Theorem 3.7]. However an ambiguity remains concerning the character obtained: whenever the center of G is not connected, there are several Gel'fand–Graev characters. The original aim of the two parts of this paper was to resolve this ambiguity. For this, we needed to study more deeply the structure of the endomorphism algebra of an induced cuspidal character sheaf; for instance, we wanted to follow the action of a Frobenius endomorphism through this algebra.

This led us to this first part, in which we develop another approach for computing explicitly this endomorphism algebra. One of the main goals is to construct another isomorphism between this endomorphism algebra and the group algebra of the relative Weyl group involved (one has already been constructed by Lusztig [14, Theorem 9.2]). In the case where G is symplectic or special orthogonal, this new isomorphism was constructed and computed explicitly by Waldspurger [23, § VIII.8].

By comparing the isomorphisms, we get some immediate consequences for finite reductive groups. The results of this part are valid for any cuspidal local system supported by a unipotent class and may perhaps be useful for computing values of characters at unipotent elements.

In the second forthcoming part, we will restrict our attention to the case where the cuspidal local system is supported by the regular unipotent class. We are then able to compute explicitly the generalized Springer correspondence through this new isomorphism. This result is valid only for p good. As an application of these (sometimes tedious) computations, we get the desired more precise version of the theorem of Digne, Lehrer and Michel. However this final result is valid only when the cardinality of the finite field is large enough.

Introduction to the first part

Let G be a connected reductive group defined over an algebraically closed field $\mathbb{F}$, let L be a Levi subgroup of a parabolic subgroup $P = L \cdot V$ of G, let C be a unipotent class of L, and let $v \in C$. We first explain how the action of the finite group $W_G(L,C) = N_G(L,C)/L$ on some varieties introduced by Lusztig [14, §§ 3, 4] can be extended 'by density' to some slightly bigger varieties (see Section 2). We then generalize this construction to extend the action of $W_G(L,v) = N_G(L,v)/C_L^\circ(v)$ to other varieties covering the previous ones (see Subsection 3.B). One of our goals is to determine the stabilizer H of an element lying over a representative u of the induced unipotent class of C (we choose u in vV).

Whenever $C_L(v)/C_L^\circ(v) \simeq C_G(v)/C_G^\circ(v)$ we have $W_G(L,v) = W_G^\circ(L,v) \times A_L(v)$ (where $A_L(v) = C_L(v)/C_L^\circ(v)$ and $W_G^\circ(L,v) = N_G(L) \cap C_G^\circ(v)/C_L^\circ(v)$). If moreover $C_G(u) \subset P$, then we show that there exists a morphism

$$\varphi_{L,v}^G \colon W_G^\circ(L,v) \to A_L(v) = C_L(v)/C_L^\circ(v)$$

such that

$$H = \{(w,a) \in W_G^\circ(L,v) \times A_L(v) \mid a = \varphi_{L,v}^G(w)\}$$

(see Subsections 3.C, 3.D). We also provide some reduction arguments to compute explicitly the morphisms $\varphi_{L,v}^G$ (see Section 4).

From Section 5 to the end, we assume that C supports a cuspidal local system $\mathcal{E}$ (we denote by ζ the character of the finite group $A_L(v)$ associated to $\mathcal{E}$). Let K denote the perverse sheaf obtained from the datum $(C,\mathcal{E})$ by parabolic induction [14, (4.1.1)], and let $\mathcal{A}$ denote its endomorphism algebra. In this case, $W_G(L,C)$ is equal to $W_G(L)$ (by [14, Theorem 9.2]) and is isomorphic to $W_G^\circ(L,v)$. Lusztig [14, Theorem 9.2] constructed a canonical isomorphism $\Theta \colon \overline{\mathbb{Q}}_\ell W_G^\circ(L,v) \to \mathcal{A}$. The aim of Section 5 is to construct another explicit isomorphism $\Theta' \colon \overline{\mathbb{Q}}_\ell W_G^\circ(L,v) \to \mathcal{A}$ using the varieties introduced in § 3. It turns out that Θ and Θ' differ by a linear character $\gamma_{L,v,\zeta}^G$ (see Corollary 6.2). To compute explicitly this linear character, one could use the characterization of Lusztig in terms of the action on the perverse sheaf K. We avoid this difficulty by using work done in earlier sections: we show that the linear character $\gamma_{L,v,\zeta}^G$ is known whenever the morphism $\varphi_{L,v}^G$ is known. (Indeed, the fact that C supports a cuspidal local system implies that $A_L(v) = A_G(v)$ and $C_G(u) \subset P$, so that the morphism $\varphi_{L,v}^G$ is defined; see Theorem 5.2.)

They are related by the formula $\gamma^G_{L,v,\zeta} = (1/\zeta(1))(\zeta \circ \varphi^G_{L,v})$ (which means that, in this case, $\varphi^G_{L,v}$ has values in the center of the character ζ).

In Section 8, we assume further that $\mathbb{F}$ is an algebraic closure of a finite field and that G is endowed with a Frobenius endomorphism. We then explain what kind of refinements may be obtained by using the previous results about characteristic functions of character sheaves.

In the case when G is special orthogonal or symplectic, the linear character $\gamma^G_{L,v,\zeta}$ has been computed directly (without using explicitly the morphism $\varphi^G_{L,v}$) by Waldspurger [23, Lemma VIII.9]. In Part II, we shall assume throughout that v is regular. Under this hypothesis, we compute explicitly the morphisms $\varphi^G_{L,v}$ even when C does not support a cuspidal local system. We then follow the method of Digne, Lehrer and Michel: from knowledge of $\gamma^G_{L,v,\zeta}$ and the explicit nature of the isomorphism Θ', we get a slightly more precise version of their theorem on Lusztig restriction of Gel'fand–Graev characters.

Our notation will be as follows.

Fields, varieties, sheaves. We fix an algebraically closed field $\mathbb{F}$ and we denote by p its characteristic. All algebraic varieties and all algebraic groups will be considered over $\mathbb{F}$. We also fix a prime ℓ different from p. Let $\overline{\mathbb{Q}}_\ell$ denote an algebraic closure of the ℓ-adic field $\mathbb{Q}_\ell$.

If X is an algebraic variety (over $\mathbb{F}$), we also denote by $\overline{\mathbb{Q}}_\ell$ the constant ℓ-adic sheaf associated to $\overline{\mathbb{Q}}_\ell$ (if necessary, we denote it by $(\overline{\mathbb{Q}}_\ell)_X$). By a constructible sheaf (respectively a local system) on X we mean a constructible $\overline{\mathbb{Q}}_\ell$-sheaf (respectively a $\overline{\mathbb{Q}}_\ell$-local system). Let $\mathcal{D}X$ denote the bounded derived category of constructible sheaves on X. If $K \in \mathcal{D}X$ and $i \in \mathbb{Z}$, we denote by $\mathcal{H}^i K$ the ith cohomology sheaf of K and if $x \in X$, then $\mathcal{H}^i_x K$ denotes the stalk at x of the constructible sheaf $\mathcal{H}^i K$. If $K \in \mathcal{D}X$, we denote by DK its Verdier dual. If $\mathcal{L}$ is a constructible sheaf on X, we identify it with its image in $\mathcal{D}X$, that is, the complex, concentrated in degree 0, whose 0th term is $\mathcal{L}$.

Let $K \in \mathcal{D}X$. We say that K is a *perverse sheaf* if the following conditions hold:

(a) $\dim \operatorname{supp} \mathcal{H}^i K \leq -i$ for all $i \in \mathbb{Z}$;

(b) $\dim \operatorname{supp} \mathcal{H}^i DK \leq -i$ for all $i \in \mathbb{Z}$.

We denote by $\mathcal{M}X$ the full subcategory of $\mathcal{D}X$ whose objects are perverse sheaves; this is an abelian category [1, (2.14), (1.3.6)].

Let Y be a locally closed, smooth, irreducible subvariety of X and let $\mathcal{L}$ be a local system on Y. We denote by $\mathrm{IC}(\overline{Y}, \mathcal{L})$ the Deligne–Goresky–MacPherson intersection cohomology complex of $\overline{Y}$ with coefficients in $\mathcal{L}$. We often identify $\mathrm{IC}(\overline{Y}, \mathcal{L})$ with its extension by zero to X; $\mathrm{IC}(\overline{Y}, \mathcal{L})[\dim Y]$ is a perverse sheaf on X.

Algebraic groups. If H is a linear algebraic group, we denote by H° the neutral component of H, by H_{uni} the closed subvariety of H consisting of unipotent elements of H, and by $Z(H)$ the center of H. If $h \in H$, then $A_H(h)$ denotes the finite group $C_H(h)/C^\circ_H(h)$, $(h)_H$ denotes the conjugacy class of h in H, and h_s (resp. h_u) denotes the semisimple (resp. unipotent) part of h. If $\mathfrak{h}$ is the Lie algebra of

H, we denote by $\mathrm{Ad}\, h\colon \mathfrak{h} \to \mathfrak{h}$ the differential at 1 of the automorphism $H \to H$, $x \mapsto {}^h x = hxh^{-1}$.

If X and Y are varieties, and if X (resp. Y) is endowed with an action of H on the right (resp. left), then, when it exists, we write $X \times_H Y$ for the quotient of $X \times Y$ by the diagonal left action of H given by $h.(x,y) = (xh^{-1}, hy)$ for all $h \in H$ and $(x,y) \in X \times Y$. If $(x,y) \in X \times Y$, and if $X \times_H Y$ exists, we denote by $x *_H y$ the image of (x,y) in $X \times_H Y$ by the canonical morphism.

Finally, if $X_1, \ldots, X_n$ are subsets or elements of H, we denote by $N_H(X_1, \ldots, X_n)$ the intersection of the normalizers $N_H(X_i)$ of X_i in H $(1 \le i \le n)$.

Reductive groups. We fix once and for all a connected reductive algebraic group G. We fix a Borel subgroup B of G and a maximal torus T of B. We denote by $X(T)$ (resp. $Y(T)$) the lattice of rational characters (resp. of one-parameter subgroups) of T. Let $W = N_G(T)/T$. Let Φ denote the root system of G relative to T and let Φ^+ (resp. Δ) denote the set of positive roots (resp. the basis) of Φ associated to B. For each root $\alpha \in \Phi$, we denote by U_α the one-parameter unipotent subgroup of G normalized by T associated to α.

We also fix a parabolic subgroup P of G and a Levi subgroup L of P. We denote by $\pi_L\colon P \to L$ the canonical projection with kernel V, the unipotent radical of P. We denote by Φ_L the root system of L relative to T; we have $\Phi_L \subset \Phi$. Finally, W_L denotes the Weyl group of L relative to T.

1 Preliminaries

1.A Centralizers. We recall two well-known results on centralizers of elements in reductive groups. The first is due to Lusztig [14, Proposition 1.2] and the second to Spaltenstein [10, Proposition 3].

Lemma 1.1 (Lusztig). (a) *Let $l \in L$ and $g \in G$. Then*

$$\dim\{xP \mid x^{-1}gx \in (l)_L \cdot V\} \le \tfrac{1}{2}(\dim C_G(g) - \dim C_L(l)).$$

(b) *If $g \in P$, then $\dim C_P(g) \ge \dim C_L(\pi_L(g))$.*

Lemma 1.2 (Spaltenstein). *If $l \in L$, then $C_V(l)$ is connected.*

We shall give several applications of these two lemmas. We first need the following technical result:

Lemma 1.3. *Let $l \in L$. Then the following are equivalent*:

(1) $C^\circ_G(l) \subset L$;

(2) $C^\circ_G(l_s) \subset L$;

(3) $C_V(l) = \{1\}$;

(4) $\dim C_V(l) = 0$.

Proof. It is clear that (b) implies (a), and that (a) implies (d). Moreover, by Lemma 1.2, (c) is equivalent to (d). It remains to prove that (c) implies (b).

For this, let s (resp. u) denote the semisimple (resp. unipotent) part of l, and assume that $C_G^\circ(s) \not\subset L$. We want to prove that $C_V(l) \neq \{1\}$. Without loss of generality, we may and do assume that $s \in T$ and $u \in U \cap L$.

Let $G' = C_G^\circ(s)$, $U' = C_U(s)$, $V' = C_V(s)$ and $B' = C_B(s)$. Then, by [5, Corollary 11.12], $u \in G'$. Moreover, by Lemma 1.2, U' and V' are connected. So $B' = T \cdot U'$ is connected. Let Φ_s denote the root system of G' relative to T, and let Φ_s^+ be the positive root system associated to the Borel subgroup B' of G'. Since $G' \not\subset L$, there exists an irreducible component Ψ of the root system Φ_s such that $\Psi \not\subset \Phi_L$. Let α_s denote the highest root of Ψ with respect to $\Psi \cap \Phi_s^+$; then $\alpha_s \notin \Phi_L$. But U_{α_s} is central in U', and so $U_{\alpha_s} \subset C_{V'}(u) = C_V(l)$. Therefore $C_V(l) \neq \{1\}$. The proof of Lemma 1.3 is now complete.

Now let $\mathcal{O}$ be the set of elements $l \in L$ such that $C_V(l) = \{1\}$. This variety was denoted by $\mathcal{U}$ in Lusztig [18, § 2].

Lemma 1.4. *The set $\mathcal{O}$ is a dense open subset of L and the map*

$$\mathcal{O} \times V \to \mathcal{O} \cdot V, \quad (l, x) \mapsto x l x^{-1}$$

is an isomorphism of varieties.

Proof. The group V acts on G by conjugation. So, by [12, Proposition 1.4], the set

$$\mathcal{N} = \{g \in G \mid \dim C_V(g) = 0\}$$

is an open subset of G. Therefore $\mathcal{N} \cap L$ is an open subset of L. Moreover $\mathcal{N} \cap L$ is not empty since any G-regular element of T belongs to $\mathcal{N} \cap L$. But, by Lemma 1.2, $\mathcal{N} \cap L = \mathcal{O}$. This proves the first assertion of the lemma.

Now let f denote the morphism defined in Lemma 1.4. Let $l \in \mathcal{O}$. Then the map $f_l \colon V \to l \cdot V$, $x \mapsto x l x^{-1}$ is injective by definition of $\mathcal{O}$, and its image is closed because it is an orbit under a unipotent group [5, Proposition 4.10]. Comparing dimensions, we obtain that f_l is bijective. As this holds for every $l \in \mathcal{O}$, f is bijective.

Moreover the variety $\mathcal{O} \cdot V \simeq \mathcal{O} \times V$ is smooth. Hence, to prove that f is an isomorphism, it is enough to prove that the differential $(df)_{(l,1)}$ is surjective for some $l \in \mathcal{O}$ (see [5, Theorems AG.17.3 and AG.18.2]).

Let $t \in T$ be a G-regular element (so that $t \in \mathcal{O}$). The tangent space to $\mathcal{O}$ at t may be identified with the Lie algebra $\mathfrak{l}$ of L via the translation by t. Writing $f(l, x) = l \cdot (l^{-1} x l x^{-1})$ for every $(l, x) \in \mathcal{O} \times V$, we may identify the differential $(df)_{(t,1)}$ with the map

$$\mathfrak{l} \oplus \mathfrak{v} \to \mathfrak{l} \oplus \mathfrak{v}, \quad l \oplus x \mapsto l \oplus (\operatorname{Ad} t^{-1} - \operatorname{Id}_{\mathfrak{v}})(x),$$

where $\mathfrak{v}$ denotes the Lie algebra of V. The bijectivity of $(df)_{(t,1)}$ follows immediately from the fact that the eigenvalues of $\operatorname{Ad} t^{-1}$ are equal to $\alpha(t)^{-1}$ for $\alpha \in \Phi^+ - \Phi_L$, and so they are different from 1 by the regularity of t.

Lemma 1.4 implies immediately the following result:

Corollary 1.5. *Let $\mathcal{S}$ be a locally closed subvariety of $\mathcal{O}$. Then the map*

$$\mathcal{S} \times V \to \mathcal{S} \cdot V, \quad (l,x) \mapsto xlx^{-1}$$

is an isomorphism of varieties.

Notation. If $\mathcal{S}$ is a locally closed subvariety of L, we denote by $\mathcal{S}_{\text{reg}}$ (or $\mathcal{S}_{\text{reg},G}$ if there is some ambiguity) the open subset $\mathcal{S} \cap \mathcal{O}$ of $\mathcal{S}$. It might be empty.

1.B The Steinberg map. Let $\nabla\colon G \to T/W$ be the Steinberg map. Recall that for $g \in G$, $\nabla(g)$ is defined to be the intersection of T with the conjugacy class of the semisimple part of g. Then ∇ is a morphism of varieties [22, § 6]. To compute the Steinberg map, we need to determine semisimple parts of elements of G. The following well-known lemma will be useful [14, (5.1)]:

Lemma 1.6. *If $g \in P$, then the semisimple part of g is V-conjugate to the semisimple part of $\pi_L(g)$. In particular, $\nabla(g) = \nabla(\pi_L(g))$.*

Proof. Let s be the semisimple part of g. Then the semisimple part of $\pi_L(g)$ is $\pi_L(s)$. However s belongs to some Levi subgroup L_0 of P. Let $x \in V$ be such that $L = {}^xL_0$. Then $xsx^{-1} \in L$ and xsx^{-1} is the semisimple part of xgx^{-1}. Therefore the semisimple part of $\pi_L(xgx^{-1}) = \pi_L(g)$ is $\pi_L(xsx^{-1}) = xsx^{-1}$.

Lemma 1.7. *$\nabla(Z(L)^\circ)$ is a closed subset of T/W and $\nabla(Z(L)^\circ_{\text{reg}})$ is an open subset of $\nabla(Z(L)^\circ)$.*

Proof. The restriction of ∇ to T is a finite quotient morphism. In particular, it is open and closed. Since it is closed, $\nabla(Z(L)^\circ)$ is a closed subset of T/W. Since it is open, $\nabla(T_{\text{reg}})$ is an open subset of T/W. But $\nabla(Z(L)^\circ_{\text{reg}}) = \nabla(T_{\text{reg}}) \cap \nabla(Z(L)^\circ)$. So Lemma 1.7 is proved.

Let $\nabla_L\colon L \to T/W_L$ denote the Steinberg map for the group L. By Lemma 1.3, we have

$$\mathcal{O} = \nabla_L^{-1}(T_{\text{reg}}/W_L). \tag{1.1}$$

1.C A family of morphisms. If $\mathcal{S}$ is a locally closed subvariety of L stable under conjugation by L, then $\mathcal{S} \cdot V$ is a locally closed subvariety of P stable under conjugation by P. We can therefore consider the quotients $G \times_L \mathcal{S}$ and $G \times_P \mathcal{S} \cdot V$. In this subsection we focus on the maps $G \times_L \mathcal{S} \to G$, $g *_L l \mapsto glg^{-1}$ and $G \times_P \mathcal{S} \cdot V \to G$, $g *_P x \mapsto gxg^{-1}$ which are well-defined morphisms of varieties.

Remark 1.8. If $\mathcal{S}$ is contained in $\mathcal{O}$, then the map $G \times_L \mathcal{S} \to G \times_P \mathcal{S} \cdot V$, $g *_L l \mapsto g *_P l$ is an isomorphism of varieties (by Corollary 1.5).

The next result is well known.

Lemma 1.9. *The map $G \times_P P \to G$, $g *_P x \mapsto gxg^{-1}$ is a projective surjective morphism of varieties. In particular, if F is a closed subvariety of P stable under conjugation by P, then the map $G *_P F \to G$, $g *_P x \mapsto gxg^{-1}$ is a projective morphism.*

Proof. Let $\tilde{X} = \{(x, gP) \in G \times G/P \mid g^{-1}xg \in P\}$. Then $\tilde{X}$ is a closed subvariety of $G \times G/P$. Moreover the variety G/P is projective. Therefore the projection $\pi\colon \tilde{X} \to G$, $(x, gP) \mapsto x$ is a projective morphism. Since every element of G is conjugate to an element of B, π is surjective.

But the maps $G \times_P P \to \tilde{X}$, $g *_P x \mapsto (gxg^{-1}, gP)$ and $\tilde{X} \to G \times_P P$, $(x, gP) \mapsto g *_P g^{-1}xg$ are mutually inverse morphisms of varieties. Through these isomorphisms, the map constructed in Lemma 1.9 may be identified with π. The proof is now complete.

The next result might be known but we have not come across it in the literature.

Lemma 1.10. *The morphisms of varieties*

$$G \times_L \mathcal{O} \to G, \quad g *_L l \mapsto glg^{-1}$$

and

$$G \times_P \mathcal{O} \cdot V \to G, \quad g *_P x \mapsto gxg^{-1}$$

are étale.

Proof. By Remark 1.8, it is sufficient to prove that the morphism

$$f\colon G \times_P \mathcal{O} \cdot V \to G, \quad g *_P x \mapsto gxg^{-1}$$

is étale. Since $G \times_P \mathcal{O} \cdot V$ and G are smooth varieties, f is étale if and only if the differential of f at any point of $G \times_P \mathcal{O} \cdot V$ is an isomorphism [9, Proposition III.10.4]. By G-equivariance of the morphism f (G acts on $G \times_P \mathcal{O} \cdot V$ by left translation on the first factor, and acts on G by conjugation), it is sufficient to prove that $(df)_{1 *_P x}$ is an isomorphism for every $x \in \mathcal{O} \cdot V$.

For this, let P^- denote the parabolic subgroup of G opposed to P (with respect to L), and let V^- denote its unipotent radical. Then $V^- \times \mathcal{O} \cdot V$ is an open neighborhood of $1 *_P x$ in $G *_P \mathcal{O} \cdot V$. Therefore it is sufficient to prove that the differential of the map

$$f^-\colon V^- \times \mathcal{O} \cdot V \to G, \quad (g, x) \mapsto gxg^{-1}$$

at $(1, x)$ is an isomorphism for every $x \in \mathcal{O} \cdot V$.

Let $\mathfrak{g}$, $\mathfrak{v}^-$, $\mathfrak{l}$ and $\mathfrak{p}$ denote the Lie algebras of G, V^-, L and P respectively. Since $\mathcal{O}$ is open in L, we may identify the tangent space to $\mathcal{O} \cdot V$ at x with $\mathfrak{p}$ (using left translation by x). Similarly, we identify the tangent space to G at x with $\mathfrak{g}$ using left translation. Using these identifications, the differential of f^- at $(1, x)$ may be identified with the map

$$\delta\colon \mathfrak{v}^- \oplus \mathfrak{p} \to \mathfrak{g}, \quad A \oplus B \mapsto (\operatorname{ad} x)^{-1}(A) - A + B.$$

For dimension reasons, we only need to prove that δ is injective.

For this, let $\lambda \in Y(T)$ be such that $L = C_G(\Im\lambda)$ and

$$P = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}\text{exists}\}.$$

For the definition of $\lim_{t\to 0}\lambda(t)$, see [6, p. 184]. We then define, for each $i \in \mathbb{Z}$,

$$\mathfrak{g}(i) = \{X \in \mathfrak{g} \mid (\operatorname{ad}\lambda(t))(X) = t^i X\}.$$

Then

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(i), \quad \mathfrak{p} = \bigoplus_{i\geq 0}\mathfrak{g}(i) \quad \text{and} \quad \mathfrak{v}^- = \bigoplus_{i<0}\mathfrak{g}(i)$$

(see [6, (5.14)]). For each $X \in \mathfrak{g}$, we denote by X_i its projection on $\mathfrak{g}(i)$.

Now let $l = \pi_L(x)$. Then it is clear that, for all $i_0 \in \mathbb{Z}$ and $X \in \mathfrak{g}(i_0)$,

$$(\operatorname{ad} x)^{-1}(X) \in (\operatorname{ad} l)^{-1}(X) + (\bigoplus_{i>i_0}\mathfrak{g}(i)). \tag{1}$$

Now let $A \oplus B \in \operatorname{Ker}\delta$, and assume that $A \neq 0$. Then there exists $i_0 < 0$ minimal among all $i < 0$ such that $A_i \neq 0$. Then, by (1), the projection of $\delta(A \oplus B)$ on $\mathfrak{g}(i_0)$ is equal to $(\operatorname{ad} l)^{-1}(A_{i_0}) - A_{i_0}$. However $\delta(A \oplus B) = 0$ and so $(\operatorname{ad} l)^{-1}(A_{i_0}) - A_{i_0} = 0$. Therefore $C_{\mathfrak{g}}(l) \not\subset \mathfrak{l}$. So $C_{\mathfrak{g}}(s) \not\subset \mathfrak{l}$, where s denotes the semisimple part of l. Now l lies in $\mathcal{O}$, and so its semisimple part s also lies in $\mathcal{O}$ by Lemma 1.3. However, by [5, Proposition 9.1 (1)], $C_{\mathfrak{g}}(s)$ is the Lie algebra of $C_G^\circ(s)$ which is contained in L by Lemma 1.3.

This contradiction shows that $A = 0$. But $0 = \delta(A, B) = (\operatorname{ad} x)^{-1}(A) - A + B$, and so $B = 0$. This completes the proof of Lemma 1.10.

1.D Isolated classes. An element $g \in G$ is said to be (G-)*isolated* if the centralizer of its semisimple part is not contained in a Levi subgroup of a proper parabolic subgroup of G.

Let L_{iso} denote the subset of L consisting of L-isolated elements, and let $T_{\mathrm{iso}} = T \cap L_{\mathrm{iso}}$. Then T_{iso} is a closed subset of T. Therefore $\nabla_L(T_{\mathrm{iso}})$ is a closed subset of T/W_L. Moreover $L_{\mathrm{iso}} = \nabla_L^{-1}(\nabla(T_{\mathrm{iso}}))$, and so L_{iso} is a closed subset of L.

From Lemma 1.9, the image of the morphism

$$G \times_P L_{\mathrm{iso}} \cdot V \to G, \quad g *_P x \mapsto gxg^{-1}$$

is a closed subvariety of G; we denote it by $X_{G,L}$. On the other hand, by Lemma 1.10, the image of the morphism

$$G \times_L \mathcal{O} \to G, \quad g *_L x \mapsto gxg^{-1}$$

is an open subset of G, which we denote by $\mathcal{O}_{G,L}$. Finally, let $Y_{G,L} = X_{G,L} \cap \mathcal{O}_{G,L}$.

Note that $W_G(L) = N_G(L)/L$ acts (on the right) on the variety $G \times_L L_{\mathrm{iso,reg}}$ in the following way. If $w \in W_G(L)$ and if $g *_L l \in G \times_L L_{\mathrm{iso,reg}}$, then

$$(g *_L l) \cdot w = g\dot{w} *_L \dot{w}^{-1} l \dot{w},$$

where $\dot{w} \in N_G(L)$ is any representative of w.

Proposition 1.11. *The map $G \times_L L_{\mathrm{iso,reg}} \to Y_{G,L}$, $g *_L l \mapsto glg^{-1}$ is a Galois étale covering with group $W_G(L)$.*

Proof. Set $\pi\colon G \times_L L_{\mathrm{iso,reg}} \to Y_{G,L}$, $g *_L l \mapsto glg^{-1}$, and $\gamma\colon G \times_L \mathcal{O} \to G$, $g *_L l \mapsto glg^{-1}$. By Lemma 1.10, γ is an étale morphism. If we prove that the square

$$\begin{array}{ccc} G \times_L L_{\mathrm{iso,reg}} & \longrightarrow & G \times_L \mathcal{O} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \gamma} \\ Y_{G,L} & \longrightarrow & G \end{array} \tag{\#}$$

is cartesian, then, by base change, we get that π is an étale morphism.

Since γ is smooth, the fibred product (scheme) of $G \times_L \mathcal{O}$ and $Y_{G,L}$ over G is reduced (because $Y_{G,L}$ is), so that it is enough to prove that $G \times_L L_{\mathrm{iso,reg}} = \gamma^{-1}(Y_{G,L})$.

Let $g *_L l \in G \times_L \mathcal{O}$ be such that $glg^{-1} \in Y_{G,L}$. Then there exist $h \in G$, $m \in L_{\mathrm{iso}}$ and $v \in V$ such that $hmvh^{-1} = glg^{-1}$. Let s, t and t' denote the semisimple parts of l, m and mv respectively. By Lemma 1.6, there exists $x \in V$ such that $t' = {}^x t$. Thus ${}^{hx}t = {}^g s$.

Since t is L-isolated, we have $Z(C_L^\circ(t))^\circ = Z(L)^\circ$. Therefore $Z(C_G^\circ(t))^\circ \subset Z(L)^\circ$. On the other hand, since $s \in \mathcal{O}$, we have $C_G^\circ(s) \subset L$, so that $Z(L)^\circ \subset Z(C_G^\circ(s))$. This proves that ${}^g Z(L)^\circ \subset {}^{hx} Z(L)^\circ$. For dimension reasons, we have ${}^g Z(L)^\circ = {}^{hx} Z(L)^\circ$, and so $Z(L)^\circ = Z(C_G^\circ(s))$. Hence l is isolated. Thus π is étale.

Now $W_G(L)$ acts freely on $G \times_L L_{\mathrm{iso,reg}}$. So the quotient morphism $G \times_L L_{\mathrm{iso,reg}} \to G \times_{N_G(L)} L_{\mathrm{iso,reg}}$ is a Galois étale covering with group $W_G(L)$. Moreover π clearly factorizes through this quotient morphism. We get an étale morphism $\pi_0\colon G \times_{N_G(L)} L_{\mathrm{iso,reg}} \to Y_{G,L}$. To finish the proof of Proposition 1.11, it is now enough to prove that π_0 is an isomorphism of varieties. Since π_0 is étale, we only need to prove that it is bijective.

Note that π_0 is clearly surjective. We prove that it is injective. Let (g, l) and (g', l') in $G \times L_{\mathrm{iso,reg}}$ be such that $glg^{-1} = g'l'g'^{-1}$. Let s and s' be the semisimple parts of l and l' respectively. Then $Z(C_L^\circ(s))^\circ = Z(L)^\circ$ since l is L-isolated. Since $l \in \mathcal{O}$, we also have $C_G^\circ(s) = C_L^\circ(s)$, and so $Z(C_G^\circ(s))^\circ = Z(L)^\circ$. Similarly $Z(C_G^\circ(s'))^\circ = Z(L)^\circ$. So ${}^{g^{-1}g'} Z(L)^\circ = Z(L)^\circ$. Therefore $g^{-1}g' \in N_G(L)$. This completes the proof.

1.E Self-opposed Levi subgroups. We are interested in the action of the relative Weyl group $W_G(L)$ on certain varieties. In this subsection (which is independent of the four previous ones), we deal with a particular case (where L is called self-opposed) for which this group is a reflection group for its action on $X(Z(L)^\circ)$. Most facts stated here will be used only in the next part.

The Levi subgroup L of the parabolic subgroup P of G is said to be *G-self-opposed* if, for every minimal parabolic subgroup Q of G containing P, we have

$|W_M(L)| = 2$, where M is the unique Levi subgroup of Q containing L. We recall in the next proposition some well-known basic properties of a G-self-opposed Levi subgroup of a parabolic subgroup of G. Most of them are due to Howlett [11] and Lusztig [13].

Proposition 1.12. *Assume that L is G-self-opposed. For every $\alpha \in \Delta - \Delta_L$, let Q_α denote the parabolic subgroup of G generated by P and $U_{-\alpha}$, M_α the unique Levi subgroup of Q_α containing L, and $s_{L,\alpha}$ the unique non-trivial element of $W_{M_\alpha}(L)$. Then the following assertions hold.*

(a) *For each $\alpha \in \Delta - \Delta_L$, $s_{L,\alpha}$ is a reflection on $X(Z(L)^\circ)$.*

(b) *$(W_G(L), (s_{L,\alpha})_{\alpha\in\Delta-\Delta_L})$ is a Coxeter system.*

(c) *If P' is a parabolic subgroup of G having L as a Levi subgroup, then P' and P are conjugate in G (and in fact, they are conjugate under $N_G(L)$).*

(d) *Let M be a Levi subgroup of a parabolic subgroup of G which contains L and let $g \in G$ be such that ${}^gL \subset M$. Then there exists $m \in M$ such that ${}^gL = {}^mL$.*

Proof. cf. [11] and [13] for (a), (b) and (c). We shall provide a general proof for (d). For this, we may assume that M is standard with respect to (T, B). Let Q denote the parabolic subgroup of G which contains P and which has M as a Levi complement. Replacing g by mg, where m is a suitable element of M, we may assume that ${}^gL$ is a standard Levi subgroup with respect to $(T, B \cap M)$.

Let P' be the standard parabolic subgroup of M having ${}^gL$ as a Levi complement. Then P and $P'V_Q$ are standard parabolic subgroups having L and ${}^gL$ as respective Levi complements (here, V_Q denotes the unipotent radical of Q). By (c), P and $P'V_Q$ are conjugate. Since they both contain B, they are equal. Hence ${}^gL = L$.

Remark 1.13. (1) It is well known that $|W_M(L)| \le 2$ for every minimal parabolic subgroup Q of G containing P and where M is the unique Levi subgroup of Q containing L.

(2) It happens frequently that $W_G(L)$ is a reflection group for its action on $X(Z(L)^\circ)$ and that L is not G-self-opposed (see [11] for the complete analysis of this question).

(3) Compare Proposition 1.12 with [7, Fact 1.1 (ii)].

A morphism $\sigma\colon \hat{G} \to G$ between two connected reductive groups is said to be *isotypic* if $\operatorname{Ker}\sigma$ is central in $\hat{G}$ and $\Im\sigma$ contains the derived subgroup of G. Note that in this case, the morphism σ induces an isomorphism between the Dynkin diagram of $\hat{G}$ and G.

The connected reductive group G is said to be *universally self-opposed* if, for every isotypic morphism $\hat{G} \to G$ such that $\hat{G}$ is a Levi subgroup of a parabolic subgroup of a connected reductive group $\hat{\Gamma}$, then $\hat{G}$ is $\hat{\Gamma}$-self-opposed.

Example 1.14. (1) A torus is universally self-opposed.

(2) If a unipotent class of G supports a cuspidal local system, then G is universally self-opposed [14, Theorem 9.2].

(3) The group $\mathrm{GL}_2(\mathbb{F})$ is $\mathrm{Sp}_4(\mathbb{F})$-self-opposed. However it is not universally self-opposed. Indeed, $\mathrm{GL}_2(\mathbb{F}) \times \mathbb{F}^\times$ is a Levi subgroup of a parabolic subgroup of $\mathrm{GL}_3(\mathbb{F})$. But it is not $\mathrm{GL}_3(\mathbb{F})$-self-opposed: one can immediately check that $N_{\mathrm{GL}_3(\mathbb{F})}(\mathrm{GL}_2(\mathbb{F}) \times \mathbb{F}^\times) = \mathrm{GL}_2(\mathbb{F}) \times \mathbb{F}^\times$.

Remark. If in Proposition 1.12 (d) we assume only that M is a connected reductive subgroup of G (and not necessarily a Levi subgroup of a parabolic subgroup), then the conclusion need not hold. Indeed, if $G = \mathrm{Sp}_4(\mathbb{F})$, if $M = \mathrm{SL}_2(\mathbb{F}) \times \mathrm{SL}_2(\mathbb{F}) \subset G$, if $L = \mathrm{SL}_2(\mathbb{F}) \times \mathbb{F}^\times \subset M$, then there exists $g \in G$ such that ${}^gL = \mathbb{F}^\times \times \mathrm{SL}_2(\mathbb{F}) \subset M$, but L and ${}^gL$ are obviously not conjugate in M. However, if $p \neq 2$, then the regular unipotent class of L supports a cuspidal local system, so that L is universally self-opposed by Example 1.14 (2).

2 Action of the relative Weyl group

2.A The set-up. *From now until the end of the paper, we denote by Σ the inverse image of an $L/Z(L)^\circ$-isolated class of $L/Z(L)^\circ$ under the canonical projection $L \to L/Z(L)^\circ$.* We also fix an element $v \in \Sigma$.

Following [14, §§ 3, 4], we consider the varieties

$$\hat{Y} = G \times \Sigma_{\mathrm{reg}},\ \tilde{Y} = G \times_L \Sigma_{\mathrm{reg}},\ \hat{X} = G \times \overline{\Sigma} V \text{ and } \tilde{X} = G \times_P \overline{\Sigma} V.$$

In these definitions, the group L (resp. P) acts on G by right translation, and acts on Σ_{reg} (resp. $\overline{\Sigma}V$) by conjugation. We also set

$$Y = \bigcup_{g \in G} g\Sigma_{\mathrm{reg}} g^{-1} \quad \text{and} \quad X = \bigcup_{g \in G} g\overline{\Sigma} V g^{-1}.$$

By Lemma 1.6, we have

$$X \subset \nabla^{-1}\big(\nabla(Z(L)^\circ)\big) \tag{2.1}$$

and

$$Y \subset \nabla^{-1}\big(\nabla(Z(L)^\circ_{\mathrm{reg}})\big). \tag{2.2}$$

Moreover X is the image of $G \times_P \overline{\Sigma}.V$ under the morphism $G \times_P P \to G$, $g *_P x \mapsto gxg^{-1}$ studied in Subsection 1.C. So, by Lemma 1.9, X is a closed irreducible subvariety of G. We set

$$\bar{\pi} \colon G \times_P \overline{\Sigma} \cdot V \to X, \quad g *_P x \mapsto gxg^{-1}.$$

It is a projective morphism of varieties.

On the other hand, $Y^+ = \bigcup_{g \in G} g(\overline{\Sigma})_{\mathrm{reg}} g^{-1}$ is in fact the intersection of X with $\nabla^{-1}\big(\nabla(Z(L)^\circ_{\mathrm{reg}})\big)$, and so it is an open subset of X (by Lemma 1.7). Moreover the inverse image of Y^+ in $G \times_L \mathcal{O}$ is equal to $\bigcup_{w \in W_G(L)} G \times_L {}^w(\overline{\Sigma})_{\mathrm{reg}}$. So, by Proposition 1.11, the map $G \times_L (\overline{\Sigma})_{\mathrm{reg}} \to Y^+$, $g *_L l \mapsto glg^{-1}$ is a Galois étale covering with group $W_G(L,\Sigma) = N_G(L,\Sigma)/L$. Since $G \times_L \Sigma_{\mathrm{reg}}$ is open in $G \times_L (\overline{\Sigma})_{\mathrm{reg}}$, its image Y under this étale morphism is open in Y^+. This proves that Y is open in X. Moreover, since the map

$$\pi \colon G \times_L \Sigma_{\mathrm{reg}} \to Y, \quad g *_L l \mapsto glg^{-1}$$

is a Galois étale covering with group $W_G(L,\Sigma)$, we get that Y is smooth (indeed, $G \times_L \Sigma_{\mathrm{reg}}$ is smooth).

Recall that $G \times_L \Sigma_{\mathrm{reg}} \to G \times_P \Sigma_{\mathrm{reg}} \cdot V$, $g *_L l \mapsto g *_P l$ is an isomorphism (see Corollary 1.5). Moreover it is clear that $\bar{\pi}^{-1}(Y) = G \times_P \Sigma_{\mathrm{reg}} \cdot V$. We summarize all of these facts in the following proposition.

Proposition 2.1 (Lusztig [14, ¶¶3.1, 3.2 and Lemma 4.3]). *With the above notation, we have the following assertions.*

(a) *X is a closed irreducible subvariety of G and Y is open in X.*

(b) *The natural map $\tilde{Y} \to \tilde{X}$, $g *_L x \mapsto g *_P x$ is an open immersion and the square*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\quad\pi\quad} & Y \\ \big\downarrow & & \big\downarrow \\ \tilde{X} & \xrightarrow{\quad\bar{\pi}\quad} & X \end{array}$$

is cartesian.

(c) *$\bar{\pi}$ (and hence π) is a projective morphism.*

(d) *π is an étale Galois covering with group $W_G(L,\Sigma)$. In particular, Y is smooth.*

Note that G acts on $\hat{Y}$, $\tilde{Y}$, $\hat{X}$ and $\tilde{X}$ by left translation on the first factor and on Y and X by conjugation. Moreover $Z(G) \cap Z(L)^\circ$ acts on the varieties $\hat{Y}$, $\tilde{Y}$, $\hat{X}$ and $\tilde{X}$ by left translation on the second factor and on Y and X by left translation. These actions of G and $Z(G) \cap Z(L)^\circ$ commute.

We have the following commutative diagram

(א)
$$\begin{array}{ccccccc} \Sigma_{\mathrm{reg}} & \xleftarrow{\quad\alpha\quad} & \hat{Y} & \xrightarrow{\quad\beta\quad} & \tilde{Y} & \xrightarrow{\quad\pi\quad} & Y \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \overline{\Sigma} & \xleftarrow{\quad\bar{\alpha}\quad} & \hat{X} & \xrightarrow{\quad\bar{\beta}\quad} & \tilde{X} & \xrightarrow{\quad\bar{\pi}\quad} & X, \end{array}$$

where α, $\bar{\alpha}$ are the canonical projections, β, $\bar{\beta}$ are the canonical quotient morphisms and the vertical maps are the natural ones. Note that the morphisms β, $\bar{\beta}$, π and $\bar{\pi}$ are $G \times (Z(G) \cap Z(L)^\circ)$-equivariant, and that the vertical maps $\hat{Y} \to \hat{X}$, $\tilde{Y} \to \tilde{X}$ and $Y \to X$ are also $G \times (Z(G) \cap Z(L)^\circ)$-equivariant.

2.B Extension of the action of $W_G(L, \Sigma)$. Although the action of $W_G(L, \Sigma)$ is defined only on $\tilde{Y}$, we will see in this subsection that it is possible to extend it to an open subset $\tilde{X}_{\min}$ of $\tilde{X}$ which, in general, strictly contains $\tilde{Y}$.

We need some preliminaries to construct this extension. If $l \in \overline{\Sigma}$, then

$$\dim C_L(l) \geq \dim C_L(v)$$

and equality holds if and only if $l \in \Sigma$. Consequently, if $g \in X$, then

$$\dim C_G(g) \geq \dim C_L(v)$$

(cf. Lemma 1.1 (b)). Moreover, if equality holds, then, by Lemma 1.1 and the previous remark, $\bar{\pi}^{-1}(g)$ is contained in

$$\tilde{X}_0 = G \times_P \Sigma \cdot V,$$

which is a smooth open subset of $\tilde{X}$.

Let

$$X_{\min} = \{g \in X \mid \dim C_G(g) = \dim C_L(v)\} \quad \text{and} \quad \tilde{X}_{\min} = \bar{\pi}^{-1}(X_{\min}).$$

By [12, Proposition 1.4], $X_{\min}$ is an open subset of X, and, by the previous discussion, $\tilde{X}_{\min} \subset \tilde{X}_0$. Moreover $Y \subset X_{\min}$ and so $\tilde{Y} \subset \tilde{X}_{\min}$. Now let $\pi_{\min} : \tilde{X}_{\min} \to X_{\min}$ denote the restriction of $\bar{\pi}$; it is a projective morphism.

Moreover, if $g \in X_{\min}$, then, by Lemma 1.1 (a), $\pi_{\min}^{-1}(g)$ is a finite set. Since the morphism $\pi_{\min}$ is projective and quasi-finite, it is finite [9, Exercise III.11.2]. We gather these facts in the next proposition.

Proposition 2.2. *With the above notation, we have the following assertions.*

(a) $X_{\min}$ *is a* $G \times (Z(G) \cap Z(L)^\circ)$*-stable open subset of* X *containing* Y*.*

(b) $\tilde{X}_{\min}$ *is a* $G \times (Z(G) \cap Z(L)^\circ)$*-stable smooth open subset of* $\tilde{X}$ *containing* $\tilde{Y}$*.*

(c) *The morphism* $\pi_{\min} : \tilde{X}_{\min} \to X_{\min}$ *is finite.*

This proposition has the following immediate consequence:

Theorem 2.3. (a) *The variety* $\tilde{X}_{\min}$ *is the normalization of* $X_{\min}$ *in the variety* $\tilde{Y}$*. Therefore there exists a unique action of the finite group* $W_G(L, \Sigma)$ *on the variety* $\tilde{X}_{\min}$ *extending its action on* $\tilde{Y}$*.*

(b) *This action is* $G \times (Z(G) \cap Z(L)^\circ)$*-equivariant, and the morphism* $\pi_{\min}$ *factorizes through the quotient* $\tilde{X}_{\min}/W_G(L, \Sigma)$*.*

(c) *If the variety* $X_{\min}$ *is normal, then* $\pi_{\min}$ *induces an isomorphism of varieties* $\tilde{X}_{\min}/W_G(L, \Sigma) \simeq X_{\min}$*.*

Notation. (1) Let $(\Sigma\cdot V)_{\min}$ denote the open subset $\Sigma\cdot V\cap X_{\min}$ of $\Sigma\cdot V$. Then $\tilde{X}_{\min}=G\times_P(\Sigma\cdot V)_{\min}$.

(2) If there is some ambiguity, we will denote by $?_L^G$ the object ? defined above (for instance, $\hat{Y}_L^G$, $\tilde{X}_{\min,L}^G$, X_L^G, $\pi_{\min,L}^G$).

2.C Unipotent classes. *From now until the end of the paper, Σ is the inverse image of a unipotent class of $L/Z(L)^\circ$.* Note that a unipotent class is isolated. Let C denote the unique unipotent class contained in Σ. From now on, the element v introduced in the previous section is chosen to be in C. Note that $\Sigma=Z(L)^\circ\cdot C\simeq Z(L)^\circ\times C$ and that $\Sigma_{\mathrm{reg}}=Z(L)^\circ_{\mathrm{reg}}\cdot C$.

Notation. (1) We denote by C^G the induced unipotent class of C from L to G, that is, the unique unipotent class C_0 of G such that $C_0\cap C\cdot V$ is dense in $C\cdot V$.

(2) If $z\in Z(L)^\circ$, the group $C_P(z)=L\cdot C_V(z)$ is connected (cf. Lemma 1.2). So it is a parabolic subgroup of $C_G^\circ(z)$ by [8, Proposition 1.11 (ii)], with unipotent radical $C_V(z)$ and Levi factor L. We denote by u_z an element of $C^{C_G^\circ(z)}\cap vC_V(z)$. We set $\tilde{u}_z=1*_P zu_z\in\tilde{X}_{\min}$.

(3) For simplicity, the unipotent element u_1 will be denoted by u, and $\tilde{u}$ stands for $\tilde{u}_1$.

Remark 2.4. Let us investigate here what are the elements of $X_{\min}$. Since $\tilde{X}_{\min}\subset\tilde{X}_0$, we only need to determine which elements of $\Sigma\cdot V$ belong to $X_{\min}$. Let $g\in\Sigma\cdot V$. Let z (resp. u') be the semisimple (resp. unipotent) part of g. By Lemma 1.6, we may assume that z belongs to $Z(L)^\circ$. Now let $G'=C_G^\circ(z)$, $P'=C_P^\circ(z)$, and $V'=C_V(z)$. Then G' is a reductive subgroup of G containing L, P' is a parabolic subgroup of G', and V' is its unipotent radical. By [5, Corollary 11.12], we have $u'\in G'$. On the other hand, $C_G^\circ(g)=C_{G'}^\circ(u')$. Now, by Lemma 1.1 (b) and by [20, Proposition II.3.2 (b) and (e)], $g\in X_{\min}$ if and only if $u'\in C^{G'}$. Hence

$$\nabla^{-1}(\nabla(z))\cap X_{\min}=(zu_z)_G \tag{2.3}$$

for every $z\in Z(L)^\circ$, and

$$X_{\min}=\bigcup_{z\in Z(L)^\circ}(zu_z)_G.$$

If $z\in Z(L)^\circ$, we denote by $H_G(L,\Sigma,z)$ the stabilizer of $\tilde{u}_z$ in $W_G(L,\Sigma)$. We first investigate the group $H_G(L,\Sigma,1)$. Recall that $C_G^\circ(u)$ is a subgroup of P, from [20, Proposition II.3.2 (e)], so that $C_G(u)/C_P(u)$ is a finite set.

Lemma 2.5. *We have $\pi_{\min}^{-1}(u)=\{g*_P u\mid g\in C_G(u)\}$. In particular,*

$$|\pi_{\min}^{-1}(u)|=|C_G(u)/C_P(u)|=|A_G(u)/A_P(u)|.$$

Proof. Clearly $\{g*_P u\mid g\in C_G(u)\}\subset\pi_{\min}^{-1}(u)$. Let $g*_P x\in\pi_{\min}^{-1}(u)$. Replacing (g,x) by (gl^{-1},lx) for some $l\in L$, we may assume that $\pi_L(x)=v$. Since $gxg^{-1}=u$, this implies that $x\in vV\cap C^G$. By [20, Proposition II.3.2 (d)], there exists $y\in P$ such that $yxy^{-1}=u$. Therefore $gy^{-1}\in C_G(u)$ and $g*_P x=gy^{-1}*_P u$.

Corollary 2.6. *If $C_G(u) \subset P$, then $\pi_{\min}^{-1}(u) = \{\tilde{u}\}$. In particular, $W_G(L,\Sigma)$ stabilizes $\tilde{u}$, that is, $H_G(L,\Sigma,1) = W_G(L,\Sigma)$.*

Next let us consider the general case. The second projection

$$\hat{X} \simeq G \times Z(L)^\circ \times \overline{C} \times V \to Z(L)^\circ$$

factors through the quotient morphism $\hat{X} \to \tilde{X}$. We denote by $\tilde{\nabla}\colon \tilde{X} \to Z(L)^\circ$ the morphism obtained after factorization. The group $W_G(L)$ acts on $Z(L)^\circ$ by conjugation, and it is easy to check that the restriction $\tilde{\nabla}_{\rm reg}\colon \tilde{Y} \to Z(L)^\circ_{\rm reg}$ of $\tilde{\nabla}$ to $\tilde{Y}$ is $W_G(L,\Sigma)$-equivariant. Hence the morphism $\tilde{\nabla}_{\min}\colon \tilde{X}_{\min} \to Z(L)^\circ$ obtained by the restriction of $\tilde{\nabla}$ is $W_G(L,\Sigma)$-equivariant.

As a consequence, we get

$$\mathrm{Stab}_{W_G(L,\Sigma)}(\tilde{g}) \subset \mathrm{Stab}_{W_G(L,\Sigma)}(\tilde{\nabla}(\tilde{g})) \tag{2.4}$$

for every $\tilde{g} \in \tilde{X}_{\min}$. Note also that $\tilde{\nabla}(\tilde{g})$ is conjugate in G to the semisimple part of $g = \pi_{\min}(\tilde{g})$ (cf. Proposition 1.6). In fact, one easily obtains a better result:

Proposition 2.7. *Let $z \in Z(L)^\circ$. Then the following assertions hold.*

(a) $H_G(L,\Sigma,z) = H_{C_G^\circ(z)}(L,\Sigma,1)$.

(b) *If $C_{C_G^\circ(z)}(u_z) \subset P$, then $H_G(L,\Sigma,z) = W_{C_G^\circ(z)}(L,\Sigma)$.*

For the proof of (a), we refer the reader to the proof of Proposition 3.4 below; the two situations are analogous and the arguments involved are very similar. We shall present them in detail only once, for Proposition 3.4, where the situation is slightly more complicated. (b) follows from (a) and Corollary 2.6.

2.D An example. In this subsection alone, we assume that $L = T$. Then $C = 1$, $\Sigma = T$, $\tilde{X} = G \times_B B$, $X = G$ and $\bar{\pi}\colon \tilde{X} \to G$ is the well-known Grothendieck map. Moreover $W_G(L,\Sigma) = W$ in this case, and $X_{\min}$ is the open subset of G consisting of regular elements. As an open subset of G, it is smooth. So the action of W on $\tilde{Y}$ extends to $\tilde{X}_{\min}$ and $\tilde{X}_{\min}/W = X_{\min}$.

Now let $\tilde{g} \in \tilde{X}_{\min}$, $g = \pi_{\min}(\tilde{g})$ and $t = \tilde{\nabla}(\tilde{g}) \in T$. We denote by $W^\circ(t)$ the Weyl group of $C_G^\circ(t)$ relative to T. The fibre $\pi_{\min}^{-1}(g)$ may be identified with the set of Borel subgroups of G containing g. Since $\tilde{X}_{\min}/W = X_{\min}$, W acts transitively on $\pi_{\min}^{-1}(g)$. However u_t is a regular unipotent element of $C_G^\circ(t)$. Therefore $C_{C_G^\circ(t)}(u_t) \subset B$. So, by Proposition 2.7, we have

$$\mathrm{Stab}_W(\tilde{g}) = W^\circ(t).$$

As a consequence, we get the well-known result

$$|\{xB \in G/B \mid g \in {}^xB\}| = |W|/|W^\circ(t)|. \tag{2.5}$$

We remark that this is not the easiest way to prove (2.5).

Example 2.8. Suppose that $G = \mathrm{GL}_2(\mathbb{F})$, that $L = T = \{\mathrm{diag}(a,b) \mid a,b \in \mathbb{F}^\times\}$ and that $\Sigma = T$. Let P^1 denote the projective line. Then

$$\tilde{X} \simeq \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [x,y] \right) \in G \times P^1 \mid [ax+by, cx+dy] = [x,y] \right\}, \quad X = G,$$

and $\pi \colon \tilde{X} \to X$ is identified with the first projection. Moreover $X_{\min}$ is the open subset of G consisting of non-central elements. We shall give a precise formula for describing the action of W on $\tilde{X}_{\min}$ in this example.

Let w denote the unique non-trivial element of W. It has order 2. We define the right action of w on $(g,[x,y]) \in \tilde{X}_{\min}$ by

$$(g,[x,y]) \cdot w = \begin{cases} (g,[bx,(d-a)x-by]) & \text{if } (bx,(d-a)x-by) \neq (0,0) \\ (g,[(a-d)y-cx,cy]) & \text{if } ((a-d)y-cx,cy) \neq (0,0), \end{cases}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One can check that, if $\tilde{X}_1$ (resp. $\tilde{X}_2$) is the open subset of $\tilde{X}$ defined by the first condition (resp. the second condition), then $\tilde{X}_1 \cup \tilde{X}_2 = \tilde{X}_{\min}$, and that the formulas given above coincide on $\tilde{X}_1 \cap \tilde{X}_2$. So we have defined a morphism of varieties. It is obviously an automorphism of order 2 and one can easily check that it extends the action of W on $\tilde{Y}$. One can also check that W acts trivially on the elements $(g,[x,y]) \in \tilde{X}_{\min}$ such that g is not semisimple, as expected from (2.5).

3 A morphism $W_G(L,\Sigma) \to A_L(v)$

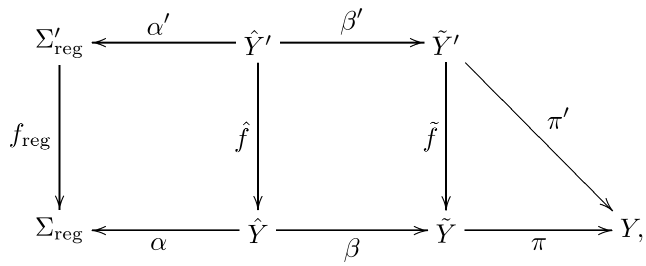
The restriction of an L-equivariant local system on C through the morphism $L/C_L^\circ(v) \to C$, $lC_L^\circ(v) \mapsto lvl^{-1}$ is constant. This fact makes this morphism interesting when one is working with character sheaves (which are equivariant intersection cohomology complexes). This morphism can be followed through diagram ($\aleph$), and it gives rise to new varieties on which the group $W_G(L,v) = N_G(L,v)/C_L^\circ(v)$ acts (note that $W_G(L,v)/A_L(v) \simeq W_G(L,\Sigma)$). Following the method of the previous section, we can extend these actions to some variety $\tilde{X}'_{\min}$ lying over $\tilde{X}_{\min}$. We show in this section that some stabilizers under this action can be described in terms of a morphism of groups $W_G(L,\Sigma) \to A_L(v)$ (under certain conditions which are fulfilled if C supports a cuspidal local system). In Section 4 we investigate the elementary properties of this morphism.

3.A Notation. Let $\Sigma' = L/C_L^\circ(v) \times Z(L)^\circ$ and $\Sigma'_{\mathrm{reg}} = L/C_L^\circ(v) \times Z(L)^\circ_{\mathrm{reg}}$. Define $f \colon \Sigma' \to \Sigma$ by $(lC_L^\circ(v), z) \mapsto lzvl^{-1} = zlvl^{-1}$. Then f is a finite surjective L-equivariant morphism (here, L acts on Σ' by left translation on the first factor). We denote by $f_{\mathrm{reg}} \colon \Sigma'_{\mathrm{reg}} \to \Sigma_{\mathrm{reg}}$ the restriction of f.

Now let

$$\hat{Y}' = G \times \Sigma'_{\mathrm{reg}} \quad \text{and} \quad \tilde{Y}' = G \times_L \Sigma'_{\mathrm{reg}} = G/C_L^\circ(v) \times Z(L)^\circ_{\mathrm{reg}}.$$

We then get a commutative diagram



where the vertical maps are induced by f_{reg}, α' is the projection on the first factor, β' is the quotient morphism, and $\pi' = \pi \circ \tilde{f}$. The group G acts on $\hat{Y}'$ and $\tilde{Y}'$ by left translation on the first factor and on Y by conjugation. The group $Z(G) \cap Z(L)^\circ$ acts on Σ'_{reg} by translation on the second factor; it induces an action on $\hat{Y}'$ and $\tilde{Y}'$. The morphisms $\hat{f}$, $\tilde{f}$, β' and π' are $G \times (Z(G) \cap Z(L)^\circ)$-equivariant. Moreover all squares in the diagram are cartesian.

Now we define

$$W_G(L, v) = N_G(L, v)/C_L^\circ(v)$$

(note that $N_G(L, v)^\circ = C_L^\circ(v)$). The group $C_L(v)$ is a normal subgroup of $N_G(L, v)$ and so $A_L(v)$ is a normal subgroup of $W_G(L, v)$. Note that

$$W_G(L, v)/A_L(v) \simeq W_G(L, \Sigma).$$

The group $N_G(L, v)$ acts freely on the right on the variety $\hat{Y}'$ in the following way: if $w \in N_G(L, v)$ and if $(g, lC_L^\circ(v), z) \in \hat{Y}'$, then

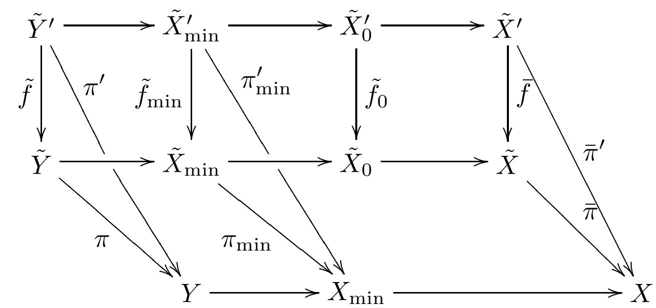
$$(g, lC_L^\circ(v), z) \cdot w = (gw, w^{-1}lwC_L^\circ(v), w^{-1}zw).$$

This induces a free right $G \times (Z(G) \cap Z(L)^\circ)$-equivariant action of $W_G(L, v)$ on $\tilde{Y}'$. Moreover the fibres of the morphism π' are $W_G(L, v)$-orbits.

Remark 3.1. If $a \in A_L(v)$ and $g *_L (lC_L^\circ(v), z) \in \tilde{Y}'$, then

$$(g *_L (lC_L^\circ(v), z)) \cdot a = g *_L (laC_L^\circ(c), z).$$

3.B Normalization. Let $\tilde{X}'$ be the normalization of the variety $\tilde{X}$ in $\tilde{Y}'$. We denote by $\bar{f}\colon \tilde{X}' \to \tilde{X}$ the corresponding morphism of varieties. Let $\tilde{X}'_0$ (resp. $\tilde{X}'_{\min}$) denote the inverse image, in $\tilde{X}'$, of the variety $\tilde{X}_0$ (resp. $\tilde{X}_{\min}$). We denote by $\bar{f}_0\colon \tilde{X}'_0 \to \tilde{X}_0$ (resp. $\bar{f}_{\min}\colon \tilde{X}'_{\min} \to \tilde{X}_{\min}$) the restriction of $\bar{f}$ to $\tilde{X}'_0$ (resp. $\tilde{X}'_{\min}$). Then $\tilde{X}'_0$ (resp. $\tilde{X}'_{\min}$) is the normalization of $\tilde{X}_0$ (resp. $\tilde{X}_{\min}$) in $\tilde{Y}'$. We summarize the notation in the following commutative diagram.



In this diagram all horizontal maps are open immersions and all squares are cartesian. Since $\tilde{X}_{\min}$ is the normalization of $X_{\min}$ in $\tilde{Y}$, we obtain the following result.

Theorem 3.2. (a) *The variety* $\tilde{X}'_{\min}$ *is the normalization of* $X_{\min}$ *in* $\tilde{Y}'$. *Therefore the action of* $W_G(L,v)$ *on* $\tilde{Y}'$ *extends uniquely to an action of* $W_G(L,v)$ *on* $\tilde{X}'_{\min}$.

(b) $\tilde{X}'$ *inherits from* $\tilde{Y}'$ *an action of* $G \times (Z(G) \cap Z(L)^\circ)$, *and this action commutes with the action of* $W_G(L,v)$ *on* $\tilde{X}_{\min}$.

Remark 3.3. We do not know how to determine the variety $\tilde{X}'$ in general. However it is possible to give an explicit description of $\tilde{X}'_0$, as follows. The parabolic subgroup P acts on $\Sigma' \times V$ by the following action: if $l, l_0 \in L$, $x, x_0 \in V$, and $z_0 \in Z(L)^\circ$, then

$$^{lx}(l_0 C_L^\circ(v), z_0, x_0) = (l l_0 C_L^\circ(v), z_0, l({}^{l_0 z_0^{-1} v^{-1} l_0^{-1}} x) x_0 x^{-1} l^{-1}).$$

It is easy to check that this defines an action of P. Moreover the morphism

$$f \times \mathrm{Id}_V : \Sigma' \times V \to \Sigma V, \quad (l_0 C_L^\circ(v), z_0, x_0) \mapsto l_0 z_0 v l_0^{-1} x_0$$

induced by f is P-equivariant.

By Corollary 1.5, $G \times_L \Sigma'_{\mathrm{reg}} \simeq G \times_P (\Sigma'_{\mathrm{reg}} \times V)$ is an open subset of $G \times_P (\Sigma' \times V)$ isomorphic to $\tilde{Y}'$. Moreover the morphism

$$G \times_P (\Sigma' \times V) \to \tilde{X} = G \times_P \Sigma V$$

induced by f is finite, as can be checked by restriction to an open subset of the form $gV^-P \times_P (\Sigma' \times V)$, where V^- is the unipotent radical of the opposite parabolic subgroup P^- of P with respect to L. Finally, by the same argument, $G \times_P (\Sigma' \times V)$ is smooth. Hence

$$\tilde{X}'_0 = G \times_P (\Sigma' \times V). \tag{3.1}$$

Since $\tilde{X}'$ is the normalization of $\tilde{X}$ in $\tilde{Y}'$, it inherits an action of the group $A_L(v)$. It is easy to describe this action on $\tilde{X}'_0$ using (3.1). It acts on $\tilde{X}'_0$ by right translation on the factor $L/C_L^\circ(v)$ of Σ'. This is a free action and the fibres of $\tilde{f}_0$ are $A_L(v)$-orbits.

We denote by $(\Sigma' \times V)_{\min}$ the inverse image, under $f \times \mathrm{Id}_V$, of the open subset $(\Sigma V)_{\min}$ of ΣV. Then $\tilde{X}'_{\min} = G \times_P (\Sigma' \times V)_{\min}$. The action of $W_G(L,v)$ is quite mysterious, but the action of its subgroup $A_L(v)$ is understandable. It is obtained by restriction from its action on $\tilde{X}'_0$ which is described at the end of Remark 3.3.

3.C Stabilizers. For $z \in Z(L)^\circ$, let $\tilde{u}'_z = 1 *_P (C^\circ_L(v), z, v^{-1}u_z) \in \tilde{X}'_{\min}$. Recall that $u_z \in vC_V(z) \cap C^{C^\circ_G(z)}$. Note that $\tilde{f}'_{\min}(\tilde{u}'_z) = \tilde{u}_z$, so that $\pi'_{\min}(\tilde{u}'_z) = u_z$. For simplicity, we write $\tilde{u}'$ for $\tilde{u}'_1$. The stabilizer of $\tilde{u}'_z$ in $W_G(L,v)$ is denoted by $H_G(L,v,z)$. The goal of this subsection is to obtain information about these stabilizers.

The first result comes from the fact that $A_L(v)$ acts freely on $\tilde{X}'_{\min}$:

$$H_G(L,v,z) \cap A_L(v) = \{1\}. \tag{3.2}$$

The second is analogous to Proposition 2.7; it may be viewed as a kind of Jordan decomposition.

Proposition 3.4. *If $z \in Z(L)^\circ$, then $H_G(L,v,z) = H_{C^\circ_G(z)}(L,v,1)$.*

Proof. Let $\tilde{\nabla}'$ denote the composite morphism of varieties $\tilde{X}' \xrightarrow{\bar{f}} \tilde{X} \xrightarrow{\tilde{\nabla}} Z(L)^\circ$, and let $\tilde{\nabla}'_{\min} \colon \tilde{X}'_{\min} \to Z(L)^\circ$ denote the restriction of $\tilde{\nabla}'$. Then $\tilde{\nabla}'_{\min}$ is a $W_G(L,v)$-equivariant morphism (as can be verified by restriction to $\tilde{Y}'$). Therefore $H_G(L,v,z)$ is contained in $\mathcal{W}_z = W_{C_G(z)}(L,v)$.

Let $A_z = \{t \in Z(L)^\circ \mid C^\circ_G(t) \subset C^\circ_G(z)\}$. Then A_z is an open subset of $Z(L)^\circ$ containing z and $Z(L)^\circ_{\mathrm{reg}}$. Now let $\Sigma_z = A_z \cdot C$ and let $\Sigma'_z = L/C^\circ_L(v) \times A_z$. Then

$$\tilde{X}'_z = G \times_P (\Sigma'_z \times V)^G_{\min,L}$$

is an open subset of $\tilde{X}_{\min}$ containing $\tilde{u}_z$, and it is stable under the action of $\mathcal{W}_z$, since A_z is and since $\tilde{X}'_z = \tilde{\nabla}'^{-1}_{\min}(A_z)$. Now let

$$\tilde{X}'(z) = C_G(z) \times_{C_P(z)} (\Sigma'_z \times C_V(z))^{C^\circ_G(z)}_{\min,L}.$$

The natural morphism $\tilde{X}'(z) \to \tilde{X}'_z$ is injective and $\mathcal{W}_z$-equivariant. This proves that the stabilizer $H_G(L,v,z)$ is equal to the stabilizer of $1 *_{C_P(z)} (C^\circ_L(v), z, v^{-1}u_z) \in \tilde{X}'(z)$ in $\mathcal{W}_z$. But this stabilizer must stabilize the connected component of $1 *_{C_P(z)} (C^\circ_L(v), z, v^{-1}u_z)$, which is $C^\circ_G(z) \times_{C_P(z)} (\Sigma'_z \times V)^{C^\circ_G(z)}_{\min,L}$ (because $C_P(z)$ is connected). Hence it is contained in $\mathcal{W}^\circ_z = W_{C^\circ_G(z)}(L,v)$ and so is equal to $H_{C^\circ_G(z)}(L,v,z)$ because this latter variety is an open subset of $(\tilde{X}'_{\min})^{C^\circ_G(z)}_L$.

Now the action of $\mathcal{W}^\circ_z$ on $(\tilde{X}'_{\min})^{C^\circ_G(z)}_L$ commutes with the translation by z. Hence $H_{C^\circ_G(z)}(L,v,z) = H_{C^\circ_G(z)}(L,v,1)$.

Remark. It is perhaps surprising that $C_G(z)$ is not necessarily connected. However one can check directly that the previous constructions ($\hat{Y}$, $\tilde{X}_{\min}$, $W_G(L,v)$, ...) remain valid if G is not connected, provided that the parabolic subgroup P of G is connected.

Proposition 3.4 shows that in order to determine the stabilizers $H_G(L,v,z)$ it is necessary and sufficient to compute the stabilizer $H_G(L,v,1)$. However we can only prove a satisfactory result when $C_G(u) \subset P$.

Proposition 3.5. *If $C_G(u) \subset P$, then the following assertions hold.*

(a) $\pi'^{-1}_{\min}(u)$ *is the $A_L(v)$-orbit of $\tilde{u}'$. In particular, $|\pi'^{-1}_{\min}(u)| = |A_L(v)|$.*
(b) $W_G(L,v) = A_L(v) \rtimes H_G(L,v,1)$.

Proof. (a) follows immediately from Corollary 2.6: indeed, $\pi'^{-1}_{\min}(u) = \tilde{f}^{-1}(\tilde{u})$. By (a), $A_L(v)$ acts freely and transitively on $\pi'^{-1}_{\min}(u)$, and so $W_G(L,v)$ acts transitively on $\pi'^{-1}_{\min}(u)$. (b) follows from this remark and from (3.2).

3.D Further investigations. The group $C^\circ_G(v) \cap L = C_{C^\circ_G(v)}(Z(L)^\circ)$ is connected, being the centralizer of a torus in a connected group [5, Corollary 11.12]. Therefore we have the well-known equality

$$C^\circ_G(v) \cap L = C^\circ_L(v). \tag{3.3}$$

Thus the natural morphism $C_L(v) \hookrightarrow C_G(v)$ induces an injective morphism

$$A_L(v) \hookrightarrow A_G(v). \tag{3.4}$$

Example 3.6. Let $G \simeq \mathrm{Sp}_4(\mathbb{F})$, $L \simeq \mathrm{GL}_2(\mathbb{F})$ and v be a regular unipotent element of L. Then $A_L(v) = \{1\}$ and $|A_G(v)| = 2$. This shows that the morphism (3.4) is in general not surjective.

Let $W^\circ_G(L,v) = N_G(L,v) \cap C^\circ_G(v)/C^\circ_L(v)$. Since $C^\circ_G(v) \cap L = C^\circ_L(v)$, we have $W^\circ_G(L,v) \cap A_L(v) = 1$. Moreover, since $W^\circ_G(L,v)$ and $A_L(v)$ are normal subgroups of $W_G(L,v)$, $W^\circ_G(L,v) \times A_L(v)$ is naturally a subgroup of $W_G(L,v)$. This discussion has the following immediate consequence:

Lemma 3.7. *If $A_L(v) = A_G(v)$, then $W_G(L,v) = W^\circ_G(L,v) \times A_L(v)$, and so $W^\circ_G(L,v) \simeq W_G(L,\Sigma)$.*

Corollary 3.8. *Assume that $C_G(u) \subset P$, and that $A_L(v) = A_G(v)$. Then there exists a unique morphism of groups $\varphi^G_{L,v} \colon W^\circ_G(L,v) \to A_L(v)$ such that*

$$H_G(L,v,1) = \{(w,a) \in W^\circ_G(L,v) \times A_L(v) \mid a = \varphi^G_{L,v}(w)\}.$$

Proof. This follows from Proposition 3.5 (b) and from Lemma 3.7.

Recall that the unipotent element v of L is said to be *distinguished in L* if v is not contained in a Levi subgroup of a proper parabolic subgroup of L.

Corollary 3.9. *Assume that v is distinguished in L, that $C_G(u) \subset P$, that $A_L(v) = A_G(v)$, and that $|A_L(v)|$ is odd. Then the morphism $\varphi^G_{L,v}$ is trivial and $H_G(L,v,1) = W^\circ_G(L,v)$.*

Proof. If v is distinguished in L, then $Z(L)^\circ$ is a maximal torus of $C_G^\circ(v)$. So $W_G^\circ(L,v)$ is the Weyl group of $C_G^\circ(v)$ relative to $Z(L)^\circ$. This shows that $W_G^\circ(L,v)$ is generated by elements of order 2. So, since $|A_L(v)|$ is odd, the morphism $\varphi_{L,v}^G\colon W_G^\circ(L,v) \to A_L(v)$ is trivial. Therefore, by Corollary 3.8, $H_G(L,v,1) = W_G^\circ(L,v)$.

Example 3.10. We will see in Theorem 5.2 that, if the class C supports a cuspidal local system, then $A_L(v) = A_G(v)$ and $C_G(u) \subset P$. Consequently, the morphism $\varphi_{L,v}^G$ is then well defined.

The morphism $\varphi_{L,v}^G$ is the central object of study in this paper. In Part II, we will compute it explicitly whenever v is a regular unipotent element under some restriction on L.

3.E Separability. Let $C^{\text{ét}}$ denote the separable closure of C in $L/C_L^\circ(v)$ (under the morphism $L/C_L^\circ(v) \to C$, $l \mapsto lvl^{-1}$). Note that $C^{\text{ét}}$ is smooth. The variety $C^{\text{ét}}$ inherits from $L/C_L^\circ(v)$ the action of L by left translation, and the action of $A_L(v)$ by right translation. Thus we have a sequence of $L \times A_L(v)$-equivariant morphisms

$$L/C_L^\circ(v) \longrightarrow C^{\text{ét}} \longrightarrow C.$$

The first morphism is bijective and purely inseparable and the second is a Galois étale covering with group $A_L(v)$. We define

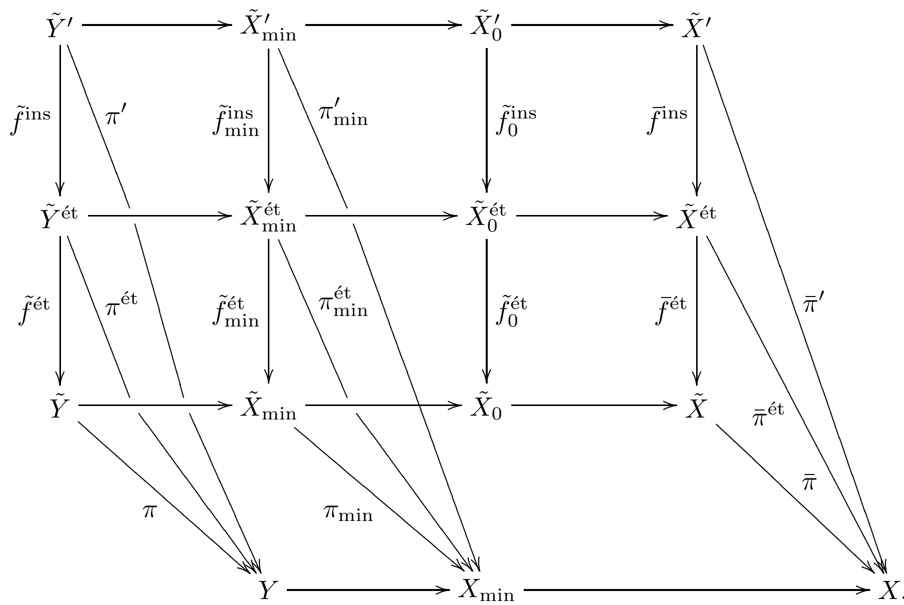
$$\Sigma^{\text{ét}} = Z(L)^\circ \times C^{\text{ét}}, \quad \Sigma^{\text{ét}}_{\text{reg}} = Z(L)^\circ_{\text{reg}} \times C^{\text{ét}}, \quad \hat{Y}^{\text{ét}} = G \times \Sigma^{\text{ét}}_{\text{reg}}, \quad \tilde{Y}^{\text{ét}} = G \times_L \Sigma^{\text{ét}}_{\text{reg}}.$$

We have a commutative diagram with cartesian squares

$$\begin{array}{ccccccc}
\Sigma'_{\text{reg}} & \xleftarrow{\alpha'} & \hat{Y}' & \xrightarrow{\beta'} & \tilde{Y}' & & \\
\downarrow f^{\text{ins}}_{\text{reg}} & & \downarrow \hat{f}^{\text{ins}} & & \downarrow \tilde{f}^{\text{ins}} & \searrow \pi' & \\
\Sigma^{\text{ét}}_{\text{reg}} & \xleftarrow{\alpha^{\text{ét}}} & \hat{Y}^{\text{ét}} & \xrightarrow{\beta^{\text{ét}}} & \tilde{Y}^{\text{ét}} & & \\
\downarrow f^{\text{ét}}_{\text{reg}} & & \downarrow \hat{f}^{\text{ét}} & & \downarrow \tilde{f}^{\text{ét}} & \searrow \pi^{\text{ét}} & \\
\Sigma_{\text{reg}} & \xleftarrow{\alpha} & \hat{Y} & \xrightarrow{\beta} & \tilde{Y} & \xrightarrow{\pi} & Y.
\end{array} \tag{3.6}$$

Here the maps $?^{\text{ét}}$ and $?^{\text{ins}}$ are induced by the maps $?$ or $?'$. Moreover the morphisms $?^{\text{ét}}$ are Galois étale coverings, and all morphisms $?^{\text{ins}}$ are bijective purely inseparable morphisms. Note that $f^{\text{ét}}_{\text{reg}}$, $\hat{f}^{\text{ét}}$ and $\tilde{f}^{\text{ét}}$ are Galois coverings with group $A_L(v)$ and $\pi^{\text{ét}}$ is a Galois covering with group $W_G(L,v)$.

By the same argument as in Remark 3.3, the group P acts on the variety $\Sigma^{\text{ét}} \times V$ and the quotient $\tilde{X}_0^{\text{ét}} = G \times_P (\Sigma^{\text{ét}} \times V)$ exists: it is the separable closure of $\tilde{X}_0$ in $\tilde{X}'_0$. If $(\Sigma^{\text{ét}} \times V)_{\min}$ denotes the inverse of $(\Sigma \cdot V)_{\min}$ under the morphism $f^{\text{ét}} \times \mathrm{Id}_V$, then $\tilde{X}^{\text{ét}}_{\min} = G \times_P (\Sigma^{\text{ét}} \times V)_{\min}$ is the normalization of $X_{\min}$ in $\tilde{Y}^{\text{ét}}$. So it inherits an action of $W_G(L,v)$ and the bijective purely inseparable morphism $\tilde{f}^{\text{ins}}_{\min} \colon \tilde{X}'_{\min} \to \tilde{X}^{\text{ét}}_{\min}$ induced by $\tilde{f}_{\min}$ is $W_G(L,v)$-equivariant. Moreover the morphism $\tilde{f}^{\text{ét}}_{\min} \colon \tilde{X}^{\text{ét}}_{\min} \to \tilde{X}_{\min}$ induced by $\tilde{f}_{\min}$ is a Galois étale covering with group $A_L(v)$. We summarize the notation in the following diagram.



Remark 3.11. For $z \in Z(L)^\circ$, we denote by $\tilde{u}^{\text{ét}}_z$ the image of $\tilde{u}'_z \in \tilde{X}'_{\min}$ in $\tilde{X}^{\text{ét}}_{\min}$ under the morphism $\tilde{f}^{\text{ins}}_{\min}$. Since $\tilde{f}^{\text{ins}}_{\min}$ is bijective and $W_G(L,v)$-equivariant, the stabilizer of $\tilde{u}^{\text{ét}}_z$ in $W_G(L,v)$ is equal to $H_G(L,v,z)$.

Example 3.12. It may happen that the variety $C^{\text{ét}}$ is different from $L/C^\circ_L(v)$, so that the construction above is not trivial. Of course this only occurs in positive characteristic. The smallest example is given by the group $L = G = \mathrm{SL}_2(\mathbb{F})$, whenever $p = 2$ and

$$v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Nevertheless, this is quite an unusual phenomenon. Indeed, if $G = \mathrm{SL}_n(\mathbb{F})$ and if p does not divide n, then the morphism $L/C^\circ_L(v) \to C$ is always étale. Also, if G is a quasisimple group of type different from A and if p is good for G, then again the morphism $L/C^\circ_L(v) \to C$ is always étale.

4 Elementary properties of $\varphi_{L,v}^G$

We shall show in Section 5 that knowledge of the morphism $\varphi_{L,v}^G$ is of fundamental importance in the description of the endomorphism algebra of an induced cuspidal character sheaf (cf. Corollary 6.7). For this reason we devote a section to gathering the properties of this morphism. These properties help to reduce the computations to small groups.

4.A Product of groups. We assume in this subsection alone that $G = G_1 \times G_2$. Let $L = L_1 \times L_2$, $v = (v_1, v_2)$, $P = P_1 \times P_2$. Then, for every $z = (z_1, z_2) \in Z(L)^\circ$, we have

$$H_G(L, v, z) = H_{G_1}(L_1, v_1, z_1) \times H_{G_2}(L_2, v_2, z_2). \tag{4.1}$$

Moreover, if $A_L(v) = A_G(v)$ and if $C_G(u) \subset P$, then

$$\varphi_{L,v}^G = \varphi_{L_1,v_1}^{G_1} \times \varphi_{L_2,v_2}^{G_2}. \tag{4.2}$$

4.B Changing the group. Let $\sigma\colon G_1 \to G$ be an isotypic morphism between connected reductive groups. We put $L_1 = \sigma^{-1}(L)$ and $P_1 = \sigma^{-1}(P)$, so that L_1 is a Levi subgroup of the parabolic subgroup P_1 of G_1. Let v_1 denote the unique unipotent element of G_1 such that $\sigma(v_1) = v$ and let C_1 be the class of v_1 in G_1. Note that $\sigma(C_1) = C$.

Lemma 4.1. (a) $\sigma(C_{L_1}(v_1))Z(G)^\circ = C_L(v)$ *and* $\sigma^{-1}(C_L(v)) = C_{L_1}(v_1)$.

(b) *The morphism* $A_{L_1}(v_1) \to A_L(v)$ *induced by* σ *is surjective.*

(c) *The morphism* $W_{G_1}(L_1, \Sigma_1) \to W_G(L, \Sigma)$ *induced by* σ *is an isomorphism.*

(d) *The morphism* $W_{G_1}(L_1, v_1) \to W_G(L, v)$ *induced by* σ *is surjective.*

(e) *The morphism* $W_{G_1}^\circ(L_1, v_1) \to W_G^\circ(L, v)$ *induced by* σ *is an isomorphism.*

(f) $C_G(u) \subset P$ *if and only if* $C_{G_1}(u_1) \subset P_1$.

(g) $A_L(v) = A_G(v)$ *if and only if* $A_{G_1}(v_1) = A_{L_1}(v_1)$.

Proof. Assertion (b) is an immediate consequence of (a). To prove (a), let $l \in C_L(v)$. Then there exists l_1 in L_1 and $z \in Z(G)^\circ$ such that $\sigma(l_1) = lz$. So there exists $x \in \operatorname{Ker}\sigma$ such that $l_1 v_1 l_1^{-1} = x v_1$. But $l_1 v_1 l_1^{-1}$ is unipotent and $\operatorname{Ker}\sigma$ consists of central semisimple elements of L_1, and so $x = 1$, that is, l_1 centralizes v_1. Hence the first assertion of (a) follows. The second follows by the same argument.

(c), (d), (e), (f) and (g) follow by similar arguments (note that $\sigma(C_{G_1}^\circ(v_1)) \cdot Z(G)^\circ = C_G^\circ(v)$).

We will denote by a subscript $?_1$ the object associated to the datum (L_1, v_1) and defined in the same way as the object ? in G (for instance, $\Sigma_1 = Z(L_1)^\circ \cdot C_1$, $\tilde{X}_1$, $\tilde{Y}_1'$).

The reader may check that $\sigma((\Sigma_1 \cdot V_1)_{\min}) \subset (\Sigma \cdot V)_{\min}$, so that σ induces a morphism $\tilde{X}_{1,\min} \to \tilde{X}_{\min}$. This morphism is $W_{G_1}(L_1, \Sigma_1)$-equivariant, as can be verified by restriction to $\tilde{Y}_1$ (here, $W_{G_1}(L_1, v_1)$ and $W_G(L, \Sigma)$ are identified via the morphism σ by Lemma 4.1 (c)). Similarly, σ induces a $W_{G_1}(L_1, v_1)$-equivariant morphism $\tilde{X}'_{1,\min} \to \tilde{X}'_{\min}$ (here, $W_{G_1}(L_1, v_1)$ acts on $\tilde{X}'_{\min}$ via the surjective morphism $W_{G_1}(L_1, v_1) \to W_G(L, v)$).

Proposition 4.2. *If $z_1 \in Z(L_1)^\circ$, then*

(a) *$H_{G_1}(L_1, \Sigma_1, z_1) = H_G(L, \Sigma, \sigma(z_1))$ and*

(b) *σ induces an isomorphism $H_{G_1}(L_1, v_1, z_1) \simeq H_G(L, v, \sigma(z_1))$.*

(c) *If moreover $C_G(u) \subset P$ and $A_L(v) = A_G(v)$, then the diagram*

$$\begin{array}{ccc} W^\circ_{G_1}(L_1, v_1) & \xrightarrow[\sim]{\sigma} & W^\circ_G(L, v) \\ \Big\downarrow \varphi^{G_1}_{L_1,v_1} & & \Big\downarrow \varphi^{G}_{L,v} \\ A_{L_1}(v_1) & \xrightarrow{\sigma} & A_L(v) \end{array}$$

is commutative.

Proof. It is clear that $H_{G_1}(L_1, \Sigma_1, z_1) \subset H_G(L, \Sigma, \sigma(z_1))$. To prove the reverse inclusion, we may and do assume that $z_1 = 1$ (by Proposition 2.7). Then if an element $w \in W_G(L, \Sigma)$ stabilizes $1 *_P u$, this proves that there exists $a \in \operatorname{Ker} \sigma$ such that $(1 *_{P_1} u_1) \cdot w = 1 *_{P_1} au_1$. However, by (2.4), $a = 1$. This proves (a).

To prove (b), we notice that $\sigma(H_{G_1}(L_1, v_1, z_1)) \subset H_G(L, v, \sigma(z_1))$ so that σ induces a morphism $H_{G_1}(L_1, v_1, z_1) \to H_G(L, v, \sigma(z_1))$. This morphism is injective by (3.2). It is surjective by the same argument as the one used in (a). The proof of (b) is complete.

By Lemma 4.1 (a), the diagram described in (c) is well defined. Its commutativity follows from (b).

4.C Parabolic restriction. In this subsection, we show that the above constructions are compatible with 'restrictions' to parabolic subgroups of G. We need some notation. Let P' denote a parabolic subgroup of G containing P, and let L' denote the unique Levi subgroup of P' containing L. Let V' denote the unipotent radical of P', and let $v' = \pi_{L'}(u)$. Then $v' \in v(V \cap L')$. We start with some elementary properties.

Proposition 4.3. (a) *$v' \in C^{L'}$.*

(b) *If $A_L(v) = A_G(v)$, then $A_L(v) = A_{L'}(v)$.*

(c) *If $C_G(u) \subset P$, then $C_{L'}(v') \subset P \cap L'$.*

Proof. By Lemma 1.1, we have $\dim C_P(u) \geq \dim C_{P \cap L'}(v') \geq \dim C_L(v)$. But

$$\dim C_P(u) = \dim C_L(v)$$

because $u \in C^G$. Therefore $\dim C_{P \cap L'}(v') = \dim C_L(v)$. This proves that

$$\dim (v')_{P \cap L'} = \dim C + \dim V \cap L'.$$

So $(v')_{P\cap L'}$ is dense in $C\cdot(V\cap L')$, that is, $v'\in C^{L'}$. Hence (a) is proved.

(b) follows from (3.3) applied to the Levi subgroup L'. Let us now prove (c). Let $m\in C'_L(v')$. We only need to prove that $m\in P$. But ${}^m u\in v'\cdot V'\cap C^G\subset vV\cap C^G$. Hence, by [20, Proposition II.3.2 (d)], there exists $x\in P$ such that ${}^m u={}^x u$. Thus $x^{-1}m\in C_G(u)$. But $C_G(u)\subset P$ by hypothesis and so $m\in P$.

Proposition 4.4. *If $C_G(u)\subset P$, then $H_G(L,v,1)\cap W_{L'}(L,v)=H_{L'}(L,v,1)$.*

Proof. By Proposition 3.5 (b), the subgroups $H_G(L,v,1)\cap W_{L'}(L,v)$ and $H_{L'}(L,v,1)$ have the same index in $W_{L'}(L,v)$ (this index is equal to $|A_L(v)|$). Consequently, it is sufficient to prove that

$$H_G(L,v,1)\cap W_{L'}(L,v)\subset H_{L'}(L,v,1). \qquad (\#)$$

Let

$$\tilde{F}'=P'\times_P(\Sigma'\times V)^G_{\min,L}.$$

Thus $\tilde{F}'$ is an irreducible closed subvariety of $\tilde{X}'$, and it is stable under the action of $W_{L'}(L,v)$ (indeed, the open subset $\tilde{O}'=P'\times_L\Sigma'_{\mathrm{reg}}$ is obviously $W_{L'}(L,v)$-stable).

By Lemma 1.1 (b), the projection $\pi_{L'}\colon P'\to L'$ sends an element of $(\Sigma\cdot V)^G_{\min,L}$ to an element of $(\Sigma\cdot(V\cap L'))^{L'}_{\min,L}$, and so it induces a map $\gamma\colon\tilde{F}'\to(\tilde{X}'_{\min})^{L'}_L$. Moreover the diagram

$$\begin{array}{ccc}\tilde{O}' & \longrightarrow & \tilde{F}'\\ \big\downarrow & & \big\downarrow{\scriptstyle\gamma}\\ (\tilde{Y}')^{L'}_L & \longrightarrow & (\tilde{X}'_{\min})^{L'}_L\end{array}$$

is commutative. The first vertical map is $W_{L'}(L,v)$-equivariant and so, by density, the second vertical map is also $W_{L'}(L,v)$-equivariant. This proves (#).

Corollary 4.5. *If $C_G(u)\subset P$ and if $A_L(v)=A_G(v)$, then*

$$\varphi^{L'}_{L,v}=\mathrm{Res}^{W^\circ_G(L,v)}_{W^\circ_{L'}(L,v)}\,\varphi^G_{L,v}.$$

Proof. By Proposition 4.3 (b) and (c), $\varphi^{L'}_{L,v}$ is well defined. So Corollary 4.5 follows from Proposition 4.4.

Remark 4.6. If G' is a connected reductive subgroup of G containing L, then it may happen that

$$\varphi^{G'}_{L,v}\neq\mathrm{Res}^{W^\circ_G(L,v)}_{W^\circ_{G'}(L,v)}\,\varphi^G_{L,v}.$$

An example is provided in Part II of this paper.

5 Endomorphism algebras of induced cuspidal character sheaves

Hypothesis and notation. *From now until the end of this first part, we assume that C supports an irreducible cuspidal* [14, Definition 2.4] *local system $\mathcal{E}$. To this local system is associated an irreducible character ζ of $A_L(v)$, via the Galois étale covering $C^{\text{ét}} \to C$. Let $\mathcal{F} = \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{E}$ ($\mathcal{F}$ is a local system on Σ) and let $\mathcal{F}_{\text{reg}}$ denote the restriction of $\mathcal{F}$ to Σ_{reg}. Let K be the perverse sheaf on G obtained from the triple (L, v, ζ) by parabolic induction* [15, (4.1.1)] *and let $\mathcal{A}$ denote its endomorphism algebra.*

In [14, Theorem 9.2], Lusztig constructed an isomorphism $\Theta\colon \overline{\mathbb{Q}}_\ell W_G(L) \to \mathcal{A}$. This isomorphism is very convenient for computing the generalized Springer correspondence. On the other hand, Lusztig's construction is canonical but not explicit. The principal aim of this paper is to construct an explicit isomorphism $\Theta'\colon \overline{\mathbb{Q}}_\ell W_G(L) \to \mathcal{A}$ by another method. It turns out that this isomorphism differs from Lusztig's one by a linear character γ of $W_G(L)$. Knowledge of γ would allow us to combine the advantages of both isomorphisms Θ and Θ'. However we are not able to determine it explicitly in general, although we can relate it to the morphism $\varphi^G_{L,v}$ defined in the previous section. Note that $\gamma = 1$ whenever $L = T$.

In this section we recall some well-known facts about cuspidal local systems, parabolic induction and endomorphism algebras. Most of these results may be found in [14] or [15]. However Theorem 5.2, which is a fundamental step for constructing the isomorphism Θ' defined above, was proved in full generality in [3]. The isomorphism Θ' will be constructed in Section 6. We will also prove in Section 6 the existence of γ and its relationship with $\varphi^G_{L,v}$. We establish in Section 7 some properties of γ which allow us to reduce its computation to the case where G is semisimple, simply connected, quasi-simple, and P is a maximal parabolic subgroup of G.

5.A Parabolic induction. For the convenience of the reader, we reproduce here the diagram (ℵ).

$$\begin{array}{ccccccc}
\Sigma_{\text{reg}} & \xleftarrow{\ \alpha\ } & \hat{Y} & \xrightarrow{\ \beta\ } & \tilde{Y} & \xrightarrow{\ \pi\ } & Y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{\Sigma} & \xleftarrow{\ \bar{\alpha}\ } & \hat{X} & \xrightarrow{\ \bar{\beta}\ } & \tilde{X} & \xrightarrow{\ \bar{\pi}\ } & X
\end{array}$$

We define $\hat{\mathcal{F}}_{\text{reg}} = \alpha^* \mathcal{F}_{\text{reg}}$: it is a local system on $\hat{Y}$. Moreover, since $\mathcal{E}$ is L-equivariant, there exists a local system $\tilde{\mathcal{F}}_{\text{reg}}$ on $\tilde{Y}$ such that $\beta^* \tilde{\mathcal{F}}_{\text{reg}} \simeq \hat{\mathcal{F}}_{\text{reg}}$. By [14, (3.2)], the morphism $\pi\colon \tilde{Y} \to Y$ is a Galois covering with Galois group $W_G(L) = N_G(L)/L$, and so $\pi_* \tilde{\mathcal{F}}_{\text{reg}} = \pi_! \tilde{\mathcal{F}}_{\text{reg}}$ (because π is finite, hence proper) is a local system on Y. We denote by K the following perverse sheaf on G:

$$K = \mathrm{IC}(\overline{Y}, \mathcal{K})[\dim Y]. \tag{5.1}$$

where $\pi_* \tilde{\mathcal{F}}_{\mathrm{reg}} = \mathcal{K}$. Recall that $\overline{Y} = X$, so that $\dim Y = \dim X$.

We shall give, following [14, § 4], an alternative description of the perverse sheaf K. Let A be the following perverse sheaf on L:

$$A = \mathrm{IC}(\overline{\Sigma}, \mathcal{F})[\dim \Sigma].$$

Note that $\overline{\Sigma} = Z(L)^\circ \overline{C}$ so that $A = \overline{\mathbb{Q}}_\ell[\dim Z(L)^\circ] \boxtimes \mathrm{IC}(\overline{C}, \mathcal{E})[\dim C]$. Since A is L-equivariant, there exists a perverse sheaf $\tilde{K}$ on $\tilde{X}$ such that

$$\bar{\alpha}^* A[\dim G + \dim V] = \bar{\beta}^* \tilde{K}[\dim P].$$

The perverse sheaf $\tilde{K}$ is in fact equal to $\mathrm{IC}(\tilde{X}, \tilde{\mathcal{F}}_{\mathrm{reg}})[\dim \tilde{X}]$. By [14, Proposition 4.5], we have

$$K = \mathrm{R}\bar{\pi}_* \tilde{K}. \tag{5.2}$$

The fact that C admits a cuspidal local system has many consequences. We gather some of them in the next two theorems.

Theorem 5.1 (Lusztig). (a) *v is a distinguished unipotent element of L.*
(b) *$N_G(L)$ stabilizes C and $\mathcal{E}$, so that $W_G(L, \Sigma) = W_G(L)$.*
(c) *L is universally self-opposed.*

Proof. cf. [14, Proposition 2.8] for (a) and [14, Theorem 9.2] for (b) and (c).

The first assertion of the next theorem was proved by Lusztig for p large enough by using the classification of cuspidal pairs [17]. In [3, Corollary to Proposition 1.1], the author gave a proof of the general case without using the classification. The second assertion is proved in [4, Theorem 3.1 (4)].

Theorem 5.2. *We have $A_L(v) = A_G(v)$ and $C_G(u) \subset P$.*

By Theorem 5.2 and Lemma 3.7, we have

$$W_G(L, v) = A_L(v) \times W_G^\circ(L, v), \tag{5.3}$$

and, by Theorem 5.1 (b), we have

$$W_G^\circ(L, v) \simeq W_G(L) = N_G(L)/L. \tag{5.4}$$

5.B Lusztig's description of $\mathcal{A}$. For each w in $W_G^\circ(L,v)$, we choose a representative $\dot{w}$ of w in $N_G(L)\cap C_G^\circ(v)$. By Theorem 5.1 (b), the local systems $\mathcal{F}$ and $(\operatorname{int}\dot{w})^*\mathcal{F}$ are isomorphic. Let θ_w denote an isomorphism of L-equivariant local systems $\mathcal{F} \to (\operatorname{int}\dot{w})^*\mathcal{F}$. Then θ_w induces an isomorphism $\tilde{\theta}_w \colon \tilde{\mathcal{F}}_{\mathrm{reg}} \to \gamma_w^*\tilde{\mathcal{F}}_{\mathrm{reg}}$, where $\gamma_w \colon \tilde{Y} \to \tilde{Y}$, $(g,xL) \mapsto (g,x\dot{w}^{-1}L)$ (cf. [14, proof of Proposition 3.5]). Since $\pi_*\gamma_w^* = \pi_*$, $\pi_*\tilde{\theta}_w$ is an automorphism of $\mathcal{K}$. By applying the functor $\mathrm{IC}(X,.)[\dim Y]$, $\pi_*\tilde{\theta}_w$ induces an automorphism Θ_w of K. The automorphism Θ_w, as well as θ_w, is defined up to multiplication by an element of $\overline{\mathbb{Q}}_\ell^{\times}$.

By [14, Proposition 3.5], $(\Theta_w)_{w\in W_G^\circ(L,v)}$ is a basis of the endomorphism algebra $\mathcal{A}$ of K; moreover, by [14, Remark 3.6], there exists a family $(a_{w,w'})_{w,w'\in W_G^\circ(L,v)}$ of elements of $\overline{\mathbb{Q}}_\ell^{\times}$ such that $\Theta_w\Theta_{w'} = a_{w,w'}\Theta_{ww'}$ for all w and w' in $W_G^\circ(L,v)$. Lusztig proved that it is possible to choose in a canonical way the family $(\theta_w)_{w\in W_G^\circ(L,v)}$ such that $\Theta_w\Theta_{w'} = \Theta_{ww'}$ for all w and w' in $W_G^\circ(L,v)$. The next theorem [14, Theorem 9.2] explains his construction.

Theorem 5.3 (Lusztig). *There exists a unique family of isomorphism s of local systems $(\theta_w \colon \mathcal{F} \to (\operatorname{int}\dot{w})^*\mathcal{F})_{w\in W_G^\circ(L,v)}$ such that the following condition holds: for each $w \in W_G^\circ(L,v)$, Θ_w acts as the identity on $\mathcal{H}_u^{-\dim Y}K$, where u is any element of C^G.*

In the previous theorem, the uniqueness of the family $(\theta_w)_{w\in W_G^\circ(L,v)}$ follows from the fact that $\mathcal{H}_u^{-\dim Y}K \neq 0$ for each $u \in C^G$. As a consequence, one concludes that the linear mapping

$$\Theta \colon \overline{\mathbb{Q}}_\ell W_G^\circ(L,v) \to \mathcal{A} = \operatorname{End}_{\mathcal{M}G}(K), \quad w \mapsto \Theta_w$$

is an isomorphism of algebras.

Remark. In [14, § 3], Lusztig starts with an element $w \in W_G(L)$ and a representative $\dot{w}$ of w in $N_G(L)$. Here we have slightly modified his argument using the fact that $W_G^\circ(L,v) \simeq W_G(L)$ which implies that every $w \in W_G(L)$ has a representative in $N_G(L)\cap C_G^\circ(v)$. Such a choice of the representatives will allow us to provide another useful family of isomorphisms $(\theta'_w \colon \mathcal{F} \to (\operatorname{int}\dot{w})^*\mathcal{F})_{w\in W_G^\circ(L,v)}$ (see Subsection 6.A).

If χ is an irreducible character of $W_G^\circ(L,v)$, we denote by K_χ an irreducible component of K associated to χ via the isomorphism Θ.

Corollary 5.4 (Lusztig). *For each $u \in C^G$, we have*

(a) $\mathcal{H}_u^{-\dim Y}K_1 = \mathcal{H}_u^{-\dim Y}K$, *and*

(b) $\mathcal{H}_u^{-\dim Y}K_\chi = 0$ *for every non-trivial irreducible character χ of $W_G^\circ(L,v)$.*

5.C Restriction to the open subset $X_{\min}$. The restriction $\tilde{\mathcal{K}}_0$ of $\tilde{K}[-\dim X]$ to $\tilde{X}_0$ is a local system [14, (4.4)], that is, a complex concentrated in degree 0, whose 0th term is a local system. In fact, $\tilde{\mathcal{K}}_0$ is the local system on $\tilde{X}_0$ associated to the Galois étale covering $\tilde{X}_0^{\text{ét}} \to \tilde{X}_0$ and to the character ζ of $A_L(v)$. Therefore the restriction $\tilde{\mathcal{K}}_{\min}$ of $\tilde{K}[-\dim X]$ to $\tilde{X}_{\min}$ is a local system. More precisely, it is the local system associated to the Galois étale covering $\tilde{X}_{\min}^{\text{ét}} \to \tilde{X}_{\min}$ and to the character ζ. Let $K_{\min}$ denote the restriction of $K[-\dim X]$ to $X_{\min}$. We have the following result.

Proposition 5.5. *We have $K_{\min} = \pi_{\min,*}\tilde{\mathcal{K}}_{\min}$. Therefore $K_{\min}$ is a constructible sheaf, that is, a complex concentrated in degree* 0*.*

Proof. Since $\pi_{\min}$ is finite, the functor $\pi_{\min,*}$ is exact. The proposition follows from this remark and the Proper Base Change Theorem.

6 Another isomorphism between $\mathcal{A}$ and $\overline{\mathbb{Q}}_\ell W_G^\circ(L,v)$

The aim of this section is to construct an explicit isomorphism Θ' between the endomorphism algebra $\mathcal{A}$ and the group algebra $\overline{\mathbb{Q}}_\ell W_G^\circ(L,v)$. Our strategy is the following. First, note that the endomorphism algebra $\mathcal{A}$ of K is canonically isomorphic to the endomorphism algebra of the local system $\mathcal{K}$ on Y. To this local system is associated a representation of the fundamental group $\pi_1(Y,y)$ of Y (where y is any point of Y). This representation and its endomorphism algebra are easy to describe (cf. (6.1)).

6.A Representations of the fundamental group. Let V_ζ denote an irreducible left $\overline{\mathbb{Q}}_\ell A_L(v)$-module affording the character ζ. We may, and we will, assume that

$$\mathcal{F} = (f^{\text{ét}})_*\overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Q}}_\ell A_L(v)} V_\zeta, \quad \mathcal{F}_{\text{reg}} = (f^{\text{ét}}_{\text{reg}})_*\overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Q}}_\ell A_L(v)} V_\zeta,$$

$$\hat{\mathcal{F}}_{\text{reg}} = (\hat{f}^{\text{ét}}_{\text{reg}})_*\overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Q}}_\ell A_L(v)} V_\zeta, \quad \tilde{\mathcal{F}}_{\text{reg}} = (\tilde{f}^{\text{ét}}_{\text{reg}})_*\overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Q}}_\ell A_L(v)} V_\zeta.$$

Here, V_ζ is identified with the constant sheaf with values in V_ζ. From the fourth equality, we deduce that

$$\mathcal{K} = \pi_*\tilde{\mathcal{F}}_{\text{reg}} = \pi_*^{\text{ét}}\overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Q}}_\ell W_G(L,v)} \operatorname{Ind}_{A_L(v)}^{W_G(L,v)} V_\zeta.$$

Therefore the endomorphism algebra of $\mathcal{K}$ is canonically isomorphic to the endomorphism algebra of the $\overline{\mathbb{Q}}_\ell W_G(L,v)$-module $\operatorname{Ind}_{A_L(v)}^{W_G(L,v)} V_\zeta$. However by (5.3), this endomorphism algebra is canonically isomorphic to $\overline{\mathbb{Q}}_\ell W_G^\circ(L,v)$. Since the functor $\operatorname{IC}(\overline{Y},\cdot)[\dim Y]$ is fully faithful, it induces an isomorphism

$$\Theta' \colon \overline{\mathbb{Q}}_\ell W_G^\circ(L,v) \to \mathcal{A}. \tag{6.1}$$

This isomorphism may be constructed in another way. The action of an element $\dot{w} \in N_G(L) \cap C_G^\circ(v)$ on $\Sigma_{\text{reg}}^{\text{ét}}$, $\hat{Y}^{\text{ét}}$ and $\tilde{Y}^{\text{ét}}$ commutes with the action of $A_L(v)$.

Therefore there exists an isomorphism $\theta'_w : \mathcal{F} \to (\operatorname{int}\dot{w})^*\mathcal{F}$ (resp. $\tau'_w : \tilde{\mathcal{F}}_{\mathrm{reg}} \to (\operatorname{int}\dot{w})^*\tilde{\mathcal{F}}_{\mathrm{reg}}$) which induces the identity on the stalks at zv (resp. $1 *_L zv$) for every $z \in Z(L)^\circ$ (resp. for every $z \in Z(L)^\circ_{\mathrm{reg}}$). Then

$$\beta^*\tau'_w = \alpha^*\theta'_w|_{\Sigma_{\mathrm{reg}}}. \tag{6.2}$$

Now let $\Theta'_w = \mathrm{IC}(\overline{Y}, \pi_*\tau'_w)[\dim Y] : K \xrightarrow{\sim} K$. By (6.2), there exists an element $\gamma_w \in \overline{\mathbb{Q}}_\ell^{\times}$ such that

$$\Theta(w) = \gamma_w \Theta'_w. \tag{6.3}$$

By considering the action on the stalk at $zv \in Y$, one obtains immediately the following result.

Proposition 6.1. *With the above notation, $\Theta'_w = \Theta'(w)$ for every $w \in W^\circ_G(L,v)$.*

Corollary 6.2. *There exists a linear character $\gamma^G_{L,v,\zeta}$ of the Weyl group $W^\circ_G(L,v)$ such that*

$$\Theta'(w) = \gamma^G_{L,v,\zeta}(w)\Theta(w)$$

for every $w \in W^\circ_G(L,v)$.

Proof. This follows from Theorem 5.3, from (6.3), and from Proposition 6.1.

Example 6.3. Whenever G is symplectic or orthogonal, L is quasisimple and $p \neq 2$, Waldspurger [23, § VIII.8] has considered an explicit subgroup of $W_G(L,v)$ (isomorphic to $W^\circ_G(L,v)$ by the canonical projection) constructed by *ad hoc* methods and has computed explicitly its action on the perverse sheaf K. It turns out that this subgroup coincides with $W^\circ_G(L,v)$ whenever G is symplectic, but does not whenever G is orthogonal. In other words, he computed explicitly the linear character $\gamma^G_{L,v,\zeta}$, up to the difference between our conventions. We give here his result.

We assume that $L \neq T$ (if $L = T$, then $\gamma^G_{L,v,\zeta} = 1$ by Corollary 6.9 below). First note that $W^\circ_G(L,v)$ is always a Weyl group of type C. We denote by γ the linear character of $W^\circ_G(L,v)$ defined as follows: it is the non-trivial linear character different from the sign character with value -1 on the reflection corresponding to the minimal Levi subgroup containing L having an irreducible root system (see Proposition 1.12 (a)). Then, by [23, Lemma VIII.9], we have, for $p \neq 2$ the following assertions.

(a) *If $G \simeq \mathrm{SO}_n(\mathbb{F})$ and if $L \simeq \mathrm{SO}_{k^2}(\mathbb{F}) \times (\mathbb{F}^\times)^{(n-k^2)/2}$, then $\gamma^G_{L,v,\zeta} = \gamma$.*

(b) *If $G \simeq \mathrm{Sp}_{2n}(\mathbb{F})$ and if $L \simeq \mathrm{Sp}_{k(k+1)} \times (\mathbb{F}^\times)^{n-(k(k+1)/2)}$, then $\gamma^G_{L,v,\zeta} = \gamma^k$.*

In the second part of this paper, the case where v is a regular unipotent element of L and p is good for G will be treated. If G is symplectic and L is of type $C_1 = A_1$, or if G is special orthogonal in even dimension and L is of type $D_2 = A_1 \times A_1$, then v is a regular unipotent element of L; the result given by Waldspurger [23, Lemma VIII.9] (and stated above with our convention) coincides with ours.

Remark 6.4. By the classification of cuspidal local systems in good characteristic [14, § 10–15], from Waldspurger's result [23, Lemma VIII.9] and our result whenever v is regular (see Part II), the only case which remains is when G is a spin group and v is not regular.

If χ is an irreducible character of $W_G^\circ(L,v)$, we denote by K'_χ an irreducible component of K associated to χ via the isomorphism Θ'. By Corollary 6.2, we have

$$K'_\chi = K_{\gamma^G_{L,v,\zeta}\chi}. \tag{6.4}$$

6.B Links between $\gamma^G_{L,v,\zeta}$ and $\varphi^G_{L,v}$. The following proposition is an immediate consequence of (6.4) and Corollary 5.4:

Proposition 6.5. *The linear character $\gamma^G_{L,v,\zeta}$ of $W_G^\circ(L,v)$ is the unique irreducible character γ of $W_G^\circ(L,v)$ satisfying $\mathcal{H}_u^{-\dim Y}K'_\gamma \neq 0$ for some (or any) $u \in C^G$.*

If χ is an irreducible character of $W_G^\circ(L,v)$, we denote by $K_{\min,\chi}$ (resp. $K'_{\min,\chi}$) the irreducible component of $K_{\min}$ associated to χ via the isomorphism Θ (resp. Θ'). Now let V_χ be an irreducible $\overline{\mathbb{Q}}_\ell W_G^\circ(L,v)$-module affording χ as character. Then, since $W_G(L,v)$ acts on $\tilde{X}^{\text{ét}}_{\min}$, it also acts on the constructible sheaf $(\pi^{\text{ét}}_{\min})_*\overline{\mathbb{Q}}_\ell$ and by construction we have

$$K'_{\min,\chi} = (\pi^{\text{ét}}_{\min})_*\overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Q}}_\ell W_G(L,v)} (V_\chi \otimes V_\zeta). \tag{6.5}$$

If $x \in X_{\min}$, then $((\pi^{\text{ét}}_{\min})_*\overline{\mathbb{Q}}_\ell)_x$ is isomorphic, as a right $\overline{\mathbb{Q}}_\ell W_G(L,v)$-module, to the permutation module associated to the set $H_x\backslash W_G(L,v)$ (here, H_x denotes the stabilizer, in $W_G(L,v)$, of a preimage of x in $\tilde{X}'_{\min}$). Since $u \in X_{\min}$, the stalk of $\mathcal{H}_u^{-\dim Y}K'_\chi$ at u may easily be computed from this remark and from (6.5): we have

$$\dim_{\overline{\mathbb{Q}}_\ell} \mathcal{H}_u^{-\dim Y}K'_\chi = \dim(K'_{\min,\chi})_u = \langle \mathrm{Res}^{W_G(L,v)}_{H_G(L,v,1)}(\chi\otimes\zeta), \mathbf{1}_{H_G(L,v,1)}\rangle$$

From this and from Proposition 6.5, we deduce immediately the following proposition.

Proposition 6.6. *The linear character $\gamma^G_{L,v,\zeta}$ is the unique irreducible character γ of $W_G^\circ(L,v)$ such that $\langle \mathrm{Res}^{W_G(L,v)}_{H_G(L,v,1)}(\gamma\otimes\zeta), \mathbf{1}_{H_G(L,v,1)}\rangle \neq 0$.*

Corollary 6.7. $\gamma^G_{L,v,\zeta} = (1/\zeta(1))\zeta \circ \varphi^G_{L,v}$.

Proof. Since $A_L(v) = A_G(v)$ and $C_G(u) \subset P$ (see Theorem 5.2), the morphism $\varphi^G_{L,v}\colon W_G^\circ(L,v) \to A_L(v)$ is well defined. Hence the corollary follows from Proposition 6.6 and Corollary 3.8.

Corollary 6.8. *If $|A_L(v)|$ is odd then $\gamma^G_{L,v,\zeta} = 1$.*

Corollary 6.9. $\gamma_{T,1,1}^G = 1$.

Example 6.10. If v is a *regular* unipotent element of L then $A_L(v)$ is abelian [21] and so ζ is a linear character. By Corollary 6.7 we get $\gamma_{L,v,\zeta}^G = \zeta \circ \varphi_{L,v}^G$. This case will be studied fully in Part II.

7 Elementary properties of the character $\gamma_{L,v,\zeta}^G$

7.A Products of groups. We assume in this subsection that $G = G_1 \times G_2$ where G_1 and G_2 are reductive groups. Let $L = L_1 \times L_2$, $v = (v_1, v_2)$, $C = C_1 \times C_2$ and $\zeta = \zeta_1 \otimes \zeta_2$. Then it is clear that

$$W_G^\circ(L,v) = W_{G_1}^\circ(L_1,v_1) \times W_{G_2}^\circ(L_2,v_2) \tag{7.1}$$

and that

$$\gamma_{L,v,\zeta}^G = \gamma_{L_1,v_1,\zeta_1}^{G_1} \otimes \gamma_{L_2,v_2,\zeta_2}^{G_2}. \tag{7.2}$$

7.B Changing the group. We use here the notation introduced in Subsection 4.B. In particular, we still denote by a subscript $?_1$ the object in G_1 corresponding to ? in G (e.g. L_1, $\mathcal{F}_1$, $\mathcal{F}_{\mathrm{reg},1}$, $\hat{Y}_1$, K_1, $\mathcal{A}_1$, $\Theta_1, \dots$). We have $\sigma^{-1}(Y) = Y_1$. Note that the groups $W_{G_1}^\circ(L_1,v_1)$ and $W_G^\circ(L,v)$ are isomorphic via σ by Lemma 4.1 (e).

Lemma 7.1. *The local system $\sigma^*\mathcal{E}_1$ on C_1 is cuspidal. It is associated to the irreducible character ζ_1 of $A_{L_1}(v_1)$ obtained from ζ by composing with the surjective morphism $A_{L_1}(v_1) \to A_L(v)$ (cf. Lemma* 4.1 (b)*).*

Proof. This is immediate from the alternative definition of a cuspidal local system given in terms of permutation representations [14, Introduction].

Proposition 7.2. (a) *The restriction of $\sigma^*(K)$ to Y_1 is isomorphic to K_1.*
(b) *σ induces an isomorphism $\dot{\sigma}\colon \mathcal{A}_1 \simeq \mathcal{A}$.*
(c) *The diagrams*

$$\begin{array}{ccc} \overline{\mathbb{Q}}_\ell W_{G_1}^\circ(L_1,v_1) & \xrightarrow{\Theta_1} & \mathcal{A}_1 \\ \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow{\scriptstyle\dot{\sigma}} \\ \overline{\mathbb{Q}}_\ell W_G^\circ(L,v) & \xrightarrow{\Theta} & \mathcal{A} \end{array} \quad \textit{and} \quad \begin{array}{ccc} \overline{\mathbb{Q}}_\ell W_{G_1}^\circ(L_1,v_1) & \xrightarrow{\Theta_1'} & \mathcal{A}_1 \\ \Big\downarrow{\scriptstyle\sigma} & & \Big\downarrow{\scriptstyle\dot{\sigma}} \\ \overline{\mathbb{Q}}_\ell W_G^\circ(L,v) & \xrightarrow{\Theta'} & \mathcal{A} \end{array}$$

are commutative.

Proof. (a) follows from the Proper Base Change Theorem [19, Chapter VI, Corollary 2.3] applied to the cartesian square

$$\begin{array}{ccc} \tilde{Y}_1 & \xrightarrow{\pi_1} & Y_1 \\ {\scriptstyle\tilde{\sigma}}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \tilde{Y} & \xrightarrow[\pi]{} & Y. \end{array}$$

(b) follows from Lusztig's description of $\mathcal{A}$. The commutativity of the first diagram in (c) follows from the fact that $\sigma(C_1^{G_1}) = C^G$ and from Theorem 5.3, while the commutativity of the second one follows from Proposition 6.1.

Corollary 7.3. *We have* $\gamma^G_{L,v,\zeta} \circ \sigma = \gamma^{G_1}_{L_1,v_1,\zeta_1}$.

7.C Parabolic restriction. Let Q be a parabolic subgroup of G containing P and let M be the Levi subgroup of Q containing L. From [14, Theorem 8.3 (b)] we have

Proposition 7.4. $\gamma^M_{L,v,\zeta} = \mathrm{Res}^{W_G^\circ(L,v)}_{W_M^\circ(L,v)} \gamma^G_{L,v,\zeta}$.

Remark 7.5. If G' is a connected reductive subgroup of G which contains L, then it may happen that $\gamma^{G'}_{L,v,\zeta} \neq \mathrm{Res}^{W_G^\circ(L,v)}_{W_{G'}^\circ(L,v)} \gamma^G_{L,v,\zeta}$. An example is provided by the group $G = \mathrm{Sp}_4(\mathbb{F})$, as will be shown in Part II of this paper.

Remark 7.6. Using (7.2), Proposition 7.4, Corollary 7.3 and Theorem 5.1 (b), the computation of $\gamma^G_{L,v,\zeta}$ can be essentially reduced to the following case: G is semisimple, simply connected, quasi-simple and L is a Levi subgroup of a maximal parabolic subgroup of G.

Remark. Corollary 6.7 can be used to give alternative proofs of (7.2), Proposition 7.2, and Proposition 7.4 (as consequences of (4.2), Proposition 4.2, and Proposition 4.4 respectively).

8 Introducing Frobenius

8.A Hypothesis and notation. In this section, and only in this section, we assume that $\mathbb{F}$ is an algebraic closure of a finite field. In particular, $p > 0$. We fix a power q of p and we denote by $\mathbb{F}_q$ the subfield of $\mathbb{F}$ with q elements. We assume also that G is defined over $\mathbb{F}_q$ and we denote by $F \colon G \to G$ the corresponding Frobenius endomorphism. If $g \in G^F$, we denote by $[g]$ (or $[g]_{G^F}$ if necessary) the G^F-conjugacy class of g.

We keep the notation introduced in Section 5 (L, C, v, $\mathcal{E}$, K, Θ, $\gamma^G_{L,v,\zeta}, \dots$). We assume that L is F-stable. Then, by Theorem 5.1 (e), there exists $n \in N_G(L)$ such that $F(P) = {}^nP$. By Lang's theorem, we can pick an element $g \in G$ such that $g^{-1}F(g) = n^{-1}$. Then ${}^gL$ and ${}^gP$ are F-stable. Since we are interested in F-stable subgroups of G which are conjugate to L under G, we may and we will assume that L and P are both F-stable. Without loss of generality we may also assume that B and T are F-stable.

We also assume that v and $\mathcal{E}$ are F-stable. Let $w \in W^\circ_G(L,v)$. We choose an element $g_w \in G$ such that $g_w^{-1}F(g_w) = \dot{w}^{-1}$ (recall that $\dot{w}$ is a representative of w in $N_G(L) \cap C^\circ_G(v)$). We then put

$$L_w = {}^{g_w}L, \quad v_w = {}^{g_w}v, \quad C_w = {}^{g_w}C, \quad \mathcal{E}_w = (\operatorname{ad} g_w^{-1})^*\mathcal{E} \quad \text{and } \mathcal{F}_w = (\operatorname{ad} g_w^{-1})^*\mathcal{F}.$$

Then L_w is an F-stable Levi subgroup of a parabolic subgroup of G, $v_w \in L_w^F$ is conjugate to v in G^F (because $g_w^{-1}F(g_w) \in C^\circ_G(v)$), C_w is the conjugacy class of v_w in L_w, $\mathcal{E}_w$ is an F-stable cuspidal local system on C_w and $\mathcal{F}_w = \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{E}_w$ (as a local system on $\Sigma_w = Z(L_w)^\circ \times C_w$).

8.B Two conjugacy results. In [3, Proposition 2.1], we proved the following result:

Proposition 8.1. *Let M and M' be two F-stable Levi subgroups of* (*non necessarily F-stable*) *parabolic subgroups of G which are geometrically conjugate and let u' be a unipotent element of M^F. Assume that the following conditions hold*:

(a) *u' is a distinguished element of M;*

(b) *$N_G(M)$ stabilizes the class $(u')_M$; and*

(c) *$A_M(u') = A_G(u')$.*

Then $[u']_{G^F} \cap M'$ is a single M'^F-conjugacy class.

Corollary 8.2. *If $w \in W^\circ_G(L,v)$, then $[v]_{G^F} \cap L_w^F = [v_w]_{L_w^F}$.*

Proof. This follows from Theorem 5.1 (a) and (c), from Theorem 5.2 and from Proposition 8.1.

The next result implies [15, II, 9.10.2].

Corollary 8.3. *Let $w \in W^\circ_G(L,v)$. Then $N_{G^F}(L_w)$ stabilizes $[v_w]_{L_w^F}$.*

8.C Characteristic functions. We choose once and for all an isomorphism of local systems $\varphi\colon F^*\mathcal{E} \to \mathcal{E}$ and we denote by $\mathcal{X}_{\mathcal{E},\varphi}$ the class function on L^F defined by

$$\mathcal{X}_{\mathcal{E},\varphi}(l) = \begin{cases} \operatorname{Tr}(\varphi_l, \mathcal{E}_l) & \text{if } l \in C^F, \\ 0 & \text{otherwise.} \end{cases}$$

Using the isomorphism $\Theta\colon \overline{\mathbb{Q}}_\ell W_G^\circ(L,v) \to \mathcal{A}$, Lusztig defined an isomorphism of local systems $\varphi_w\colon F^*\mathcal{E}_w \to \mathcal{E}_w$. We recall his construction [16, (9.3)]. Let $\theta_w\colon \mathcal{E} \to (\operatorname{ad}\dot{w})^*\mathcal{E}$ be the isomorphism of local systems defined in Theorem 5.3. Then θ_w induces an isomorphism of local systems

$$F^* \circ (\operatorname{ad} g_w^{-1})^*\theta_w\colon F^*\mathcal{E}_w \to (\operatorname{ad} g_w^{-1})^* \circ F^*\mathcal{E}.$$

Moreover φ induces an isomorphism

$$(\operatorname{ad} g_w^{-1})^*\varphi\colon (\operatorname{ad} g_w^{-1})^* \circ F^*\mathcal{E} \to \mathcal{E}_w.$$

Composing these two isomorphisms, we get an isomorphism

$$\varphi_w\colon F^*\mathcal{E}_w \to \mathcal{E}_w.$$

Once φ is chosen, the isomorphism φ_w depends only on the construction of θ_w. By Corollary 6.2, knowledge of θ_w is equivalent to knowledge of the linear character $\gamma_{L,v,\zeta}^G$.

If χ is an F-stable irreducible character of $W_G^\circ(L,v)$, then we denote by $\tilde{\chi}$ the *preferred extension* of χ to $W_G^\circ(L,v) \rtimes \langle F\rangle$ (the preferred extension has been defined by Lusztig [15]). The choice of φ and $\tilde{\chi}$ induces a well-defined isomorphism $\varphi_\chi\colon F^*K_\chi \xrightarrow{\sim} K_\chi$. We set, for every $g \in G^F$,

$$\mathcal{X}_{K_\chi,\varphi_\chi}(g) = \begin{cases} \displaystyle\sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}(\mathcal{H}_g^i(\varphi_\chi), \mathcal{H}_g^i K_\chi) & \text{if } g \text{ is unipotent,}\\ 0 & \text{otherwise.}\end{cases}$$

The significance of the knowledge of the characteristic functions $\mathcal{X}_{\mathcal{E}_w,\varphi_w}$ (equivalently, of the linear character $\gamma_{L,v,\zeta}^G$) is given by the following theorem of Lusztig.

Theorem 8.4 (Lusztig). *If p is almost good for G and if q is large enough, we have*

$$\mathcal{X}_{K_\chi,\varphi_\chi} = \frac{1}{|W_G^\circ(L,v)|} \sum_{w\in W_G^\circ(L,v)} \tilde{\chi}(wF) R_{L_w}^G(\mathcal{X}_{\mathcal{E}_w,\varphi_w}).$$

We conclude this section by explaining how the characteristic functions $\mathcal{X}_{\mathcal{E}_w,\varphi_w}$ can be computed explicitly once the linear character $\gamma_{L,v,\zeta}^G$ is known (see [23, § VIII] for a particular case). It follows from Lang's theorem that the set of rational conjugacy classes contained in C_w^F is in one-to-one correspondence with $H^1(F, A_L(v_w)) \simeq H^1(\dot{w}F, A_L(v)) = H^1(F, A_L(v))$ (the last equality follows from the fact that $W_G^\circ(L,v)$ acts trivially on $A_L(v)$). Let $a \in H^1(F, A_L(v))$. We denote by $\hat{a}$ a representative of a in $A_L(v)$ and by $v_{w,a}$ a representative of the rational conjugacy class contained in C_w^F parameterized by a. If $w = 1$, we denote by v_a the element $v_{w,a}$. We have $v_a \in L^F$; note that $[v_{w,a}]_{L_w^F} = [v_a]_{G^F} \cap L^F$ (cf. Corollary 8.2).

By following step by step the construction of the isomorphisms φ_w, we obtain that the link between the class functions $\mathcal{X}_{\mathcal{E},\varphi}$ and $\mathcal{X}_{\mathcal{E}_w,\varphi_w}$ is given in terms of the linear character $\gamma_{L,v,\zeta}^G$. More precisely, we get

Proposition 8.5. *Let $w \in W_G^\circ(L,v)$ and let $a \in H^1(F, A_L(v))$. Then*

$$\mathcal{X}_{\mathcal{E}_w,\varphi_w}(v_{w,a}) = \mathcal{X}_{\mathcal{E},\varphi}(v_a)\gamma_{L,v,\zeta}^G(w).$$

Assume now until the end of this section that ζ is a linear character. In this case, we have

$$\mathcal{X}_{\mathcal{E}_w,\varphi_w}(v_{w,a}) = \mathcal{X}_{\mathcal{E}_w,\varphi_w}(v_w)\zeta(\hat{a}). \tag{8.1}$$

Note that $\zeta(\hat{a})$ does not depend on the choice of $\hat{a}$ because ζ is F-stable. Hence we deduce from Proposition 8.5 the following result:

Corollary 8.6. *Assume that ζ is a linear character, and let $w \in W_G^\circ(L,v)$ and $a \in H^1(F, A_L(v))$. Then*

$$\mathcal{X}_{\mathcal{E}_w,\varphi_w}(v_{w,a}) = \mathcal{X}_{\mathcal{E},\varphi}(v)\gamma_{L,v,\zeta}^G(w)\zeta(\hat{a}).$$

Remark 8.7. In the theory of character sheaves applied to finite reductive groups, the characteristic functions $\mathcal{X}_{\mathcal{E}_w,\varphi_w}$ play a crucial role, as shown in Theorem 8.4. Corollary 8.6 shows the importance of the determining of the linear character $\gamma_{L,v,\zeta}^G$. We will show in Part II how the knowledge of the linear character $\gamma_{L,v,\zeta}^G$ for v regular and p good for L leads to an improvement of the theorem of Digne, Lehrer and Michel [7, Theorem 3.7] on Lusztig restriction of Gel'fand–Graev characters.

Remark 8.8. The characteristic function $\mathcal{X}_{\mathcal{E}_w,\varphi_w}$ depends on the choice of the isomorphism φ. Since $\mathcal{E}$ is an irreducible local system, two isomorphisms between $F^*\mathcal{E}$ and $\mathcal{E}$ differ only by a scalar. Hence the two characteristic functions that they define also differ by the same constant. This shows that the formula in Corollary 8.6 cannot be improved, because the factor $\mathcal{X}_{\mathcal{E},\varphi}(v)$ depends on the choice of the isomorphism φ: multiplying φ by a scalar we can give it any non-zero value.

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